436-202 MECHANICS 1

UNIT 2

DYNAMICS OF MACHINES

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.

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CONTENTS

INTRODUCTION.

The purpose of this text is to provide the students with the theoretical background and engineering applications of the two dimensional mechanics of the rigid body. It is divided into four chapters.

The first one, **Dynamic of particles**, deals with these engineering problems that can be modelled by means of limited number of particles. A large class of engineering problems falls into this category.

The second chapter, **Plane dynamics of a rigid body**, consists of two sections. The first one, *Kinematics*, deals with the geometry of motion of rigid bodies in terms of the inertial as well as in terms of the non-inertial system of coordinates. In the second section, entitled *Kinetics*, the equations that determine the relationship between motion and the forces acting on a rigid body are derived. This chapter provides bases for development of methods of modelling and analysis of the *plane mechanical systems*. Mechanical system usually refers to a number of rigid bodies connected to each other by means of the *kinematic constraints* or the *elastic elements*. The last two chapters are devoted to the engineering problems that can be approximated by the plane mechanical systems.

The third chapter, **Dynamic Analysis of Plane Mechanisms**, provides a systematic approach to the dynamics of a number of rigid bodies interconnected by the kinematic constraints. Formulation of the differential equations of motion as well as the determination of the interaction forces in the kinematic joints are covered.

Each chapter is supplied with several engineering problems. Solution to some of them are provided. Solution to the other problems should be produced by students during tutorials and in their own time.

Chapter 1
DYNAMICS OF PARTICLES

1.1 KINEMATICS OF A PARTICLE

To consider motion of a particle we assume the existence of so-called *absolute* (motionless) system of coordinates $X_aY_aZ_a$(see Fig. 1).

DEFINITION: *Inertial system of coordinated* **is one that does not rotate and which origin is fixed in the absolute space or moves along straight line at a constant velocity.**

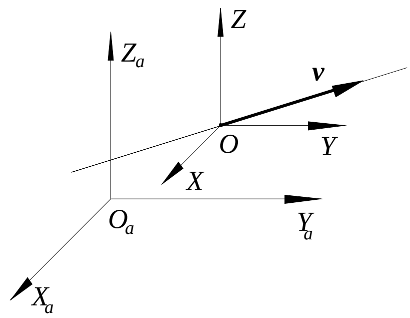


Figure 1

The inertial systems of coordinates are usually denoted by upper characters, *e.g.* XYZ, to distinguish them from *non-inertial* systems of coordinates that are usually denoted by lower characters, e.g. xyz.

1.1.1 ***Absolute linear velocity and absolute linear acceleration***

Let us assume that motion of a particle is given by a set of parametric equations 1.1 that determines the particle's coordinates at any instant of time (see Fig. 2).

$$\begin{aligned} r_X &= r_X(t) \\ r_Y &= r_Y(t) \\ r_Z &= r_Z(t) \end{aligned} \tag{1.1}$$

These coordinates represent scalar magnitude of components of so called *absolute position vector* $\mathbf{r}$ along the inertial system of coordinates XYZ.

$$\mathbf{r} = \mathbf{I}r_X(t) + \mathbf{J}r_Y(t) + \mathbf{K}r_Z(t) \tag{1.2}$$

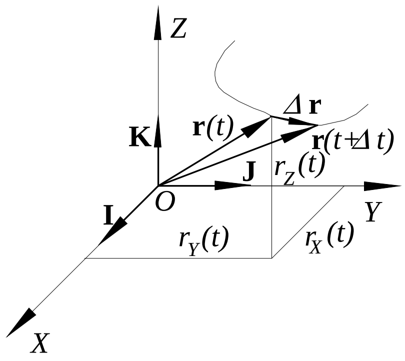


Figure 2

where $\mathbf{I}, \mathbf{J}, \mathbf{K}$ are *unit vectors* of the inertial system of coordinates XYZ. Vector of the absolute velocity, as the first derivative of the absolute position vector $\mathbf{r}$ with respect to time, is given by the following formula.

$$\mathbf{v} = \dot{\mathbf{r}} = \lim_{\Delta t \to O} \frac{\Delta \mathbf{r}}{\Delta t} = \mathbf{I} \lim_{\Delta t \to O} \frac{\Delta r_X}{\Delta t} + \mathbf{J} \lim_{\Delta t \to O} \frac{\Delta r_Y}{\Delta t} + \mathbf{K} \lim_{\Delta t \to O} \frac{\Delta r_Z}{\Delta t} = \mathbf{I}\dot{r}_X + \mathbf{J}\dot{r}_Y + \mathbf{K}\dot{r}_Z \tag{1.3}$$

Similarly, vector of the absolute acceleration is defined as the second derivative of the position vector with respect to time.

$$\mathbf{a} = \ddot{\mathbf{r}} = \lim_{\Delta t \to O} \frac{\Delta \mathbf{v}}{\Delta t} = \mathbf{I}\ddot{r}_X + \mathbf{J}\ddot{r}_Y + \mathbf{K}\ddot{r}_Z \tag{1.4}$$

Scalar magnitude of velocity v (speed) can be expressed by the following formula

$$v = |\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\dot{r}_X^2 + \dot{r}_Y^2 + \dot{r}_Z^2} \tag{1.5}$$

Scalar magnitude of acceleration is

$$a = |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{\ddot{r}_X^2 + \ddot{r}_Y^2 + \ddot{r}_Z^2} \tag{1.6}$$

The distance done by the particle in a certain interval of time $0 < \tau < t$ is given by the formula *1.7.*

$$s = \int_0^t v \, d\tau = \int_0^t \sqrt{\dot{r}_X(\tau)^2 + \dot{r}_Y(\tau)^2 + \dot{r}_Z(\tau)^2} d\tau = s(t) \tag{1.7}$$

1.1.2 ***Intrinsic system of coordinates.***

The formula 1.7 assigns to any instant of time a scalar distance s done by the point P measured along the path from a datum usually corresponding to time $t = 0$ (Fig. 3). Therefore, it is possible to express the position vector $\mathbf{r}$ as a function of the distance s.

$$\mathbf{r} = \mathbf{I}r_X(s) + \mathbf{J}r_Y(s) + \mathbf{K}r_Z(s) \tag{1.8}$$

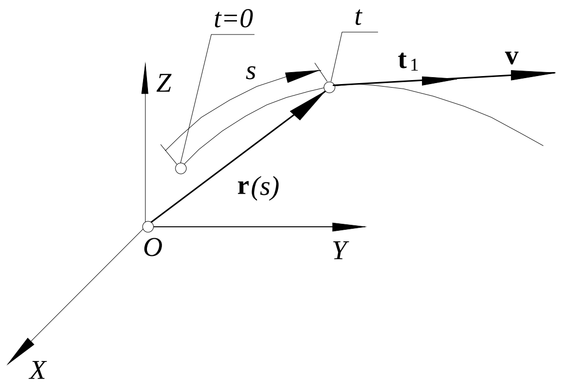


Figure 3

The vector of velocity $\mathbf{v}$ is tangential to the path. Unit vector $\mathbf{t}_1$ that possesses the direction and sense of $\mathbf{v}$ is called *tangential unit vector*.

$$\mathbf{t}_1 = \frac{\mathbf{v}}{v} = \frac{d\mathbf{r}}{dt}\frac{1}{v} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \frac{d\mathbf{r}}{ds} \tag{1.9}$$

The infinitesimal vector increment $d\mathbf{t}_1$ (see Fig. 4.) is normal to the vector $\mathbf{t}_1$ and defines the direction and sense of the *normal unit vector* $\mathbf{n}_1$. Hence

$$\mathbf{n}_1 = \frac{\frac{d\mathbf{t}_1}{dt}}{\frac{dt_1}{dt}} \tag{1.10}$$

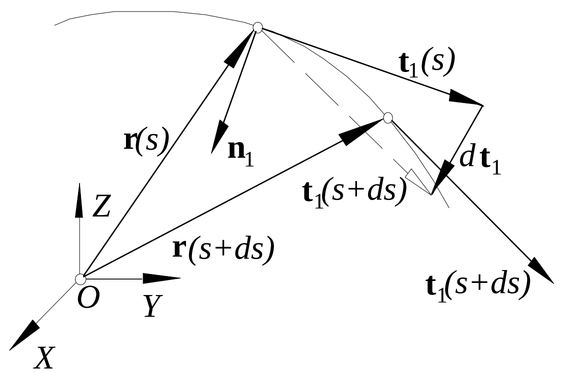


Figure 4

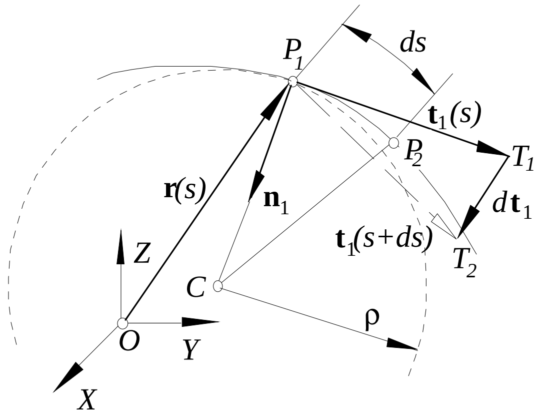


Figure 5

Vectors $\mathbf{t}_1$ and $\mathbf{n}_1$ define so-called *osculating plane* which contains the *osculating circle* of radius ρ. Since triangle CP_1P_2 is similar to $P_1T_1T_2$ (see Fig. 5)

$$\frac{dt_1}{t_1} = \frac{ds}{\rho} \tag{1.11}$$

Hence, according to Eq's.. 1.10, 1.11 and 1.9 one can obtain

$$\mathbf{n}_1 = \frac{d\mathbf{t}_1}{dt_1} = \frac{d\mathbf{t}_1\rho}{ds\ t_1} = \frac{d\mathbf{t}_1\rho}{ds} = \rho\frac{d}{ds}\left(\frac{d\mathbf{r}}{ds}\right) = \rho\frac{d^2\mathbf{r}}{ds^2} \tag{1.12}$$

Since $\mathbf{n}_1$ is an unit vector,

$$n_1 = \rho\left(\left(\frac{d^2r_X}{ds^2}\right)^2 + \left(\frac{d^2r_Y}{ds^2}\right)^2 + \left(\frac{d^2r_Z}{ds^2}\right)^2\right)^{1/2} = 1 \tag{1.13}$$

Hence

$$\rho = \frac{1}{\sqrt{\left(\frac{d^2r_X}{ds^2}\right)^2 + \left(\frac{d^2r_Y}{ds^2}\right)^2 + \left(\frac{d^2r_Z}{ds^2}\right)^2}} \tag{1.14}$$

The third unit vector, *binormal* $\mathbf{b}_1$, which together with $\mathbf{t}_1$ and $\mathbf{n}_1$ forms so-called *intrinsic system of coordinates* is defined by Eq. 1.15

$$\mathbf{b}_1 = \mathbf{t}_1 \times \mathbf{n}_1 \tag{1.15}$$

The components of the vector of acceleration $\mathbf{a}$ along the intrinsic system of coordinates are

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v\mathbf{t}_1) = \dot{v}\mathbf{t}_1 + v\dot{\mathbf{t}}_1 \tag{1.16}$$

Since according to Eq's.. 1.10 and 1.11

$$\dot{\mathbf{t}}_1 = \frac{dt_1}{dt}\mathbf{n}_1 = \frac{ds\ t_1}{\rho\ dt}\mathbf{n}_1 = \frac{1}{\rho}\frac{ds}{dt}\mathbf{n}_1 = \frac{1}{\rho}v\ \mathbf{n}_1 \tag{1.17}$$

the expression for acceleration has the following form

$$\mathbf{a} = \dot{v}\mathbf{t}_1 + \frac{1}{\rho}v^2\mathbf{n}_1 \tag{1.18}$$

The first term represents the tangential component of the absolute acceleration whereas the second term is called the normal component of the absolute acceleration.

1.1.3 *Problems*

Problem 1

Motion of a particle with respect to the inertial space is given by the following position vector

$$\mathbf{r} = \mathbf{I}a\cos\omega t + \mathbf{J}a\sin\omega t + \mathbf{K}bt \tag{1.19}$$

Determine unit vectors that are tangential ($\mathbf{t}_1$), normal ($\mathbf{n}_1$) and binormal ($\mathbf{b}_1$) to trajectory of motion of the particle as well as components of its absolute acceleration along those unit vectors.

Solution

The equation 1.19 represents motion along the screw line coiled on the cylinder of radius a as shown in Fig. 6

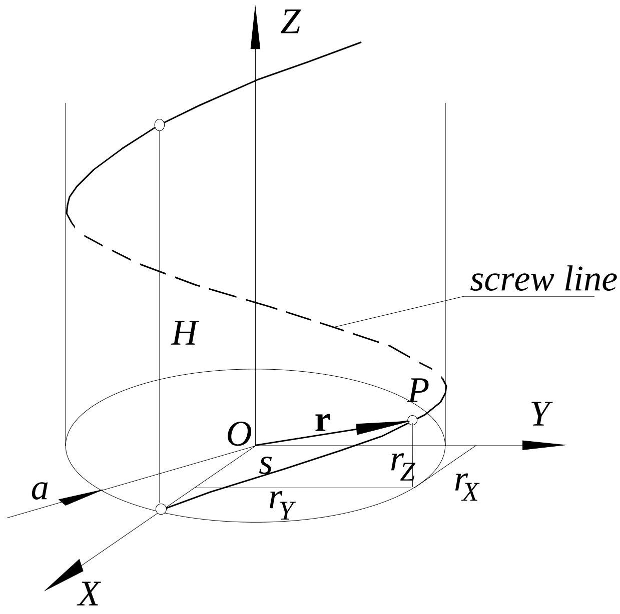


Figure 6

Its parametric equations are

$$\begin{aligned} r_X &= a\cos\omega t \\ r_Y &= a\sin\omega t \\ r_Z &= bt \end{aligned} \tag{1.20}$$

When the point P follows the screw line, its projection on the plane XY follows the circle of the radius a.

$$\begin{aligned} r_X &= a\cos\omega t \\ r_Y &= a\sin\omega t \end{aligned} \tag{1.21}$$

The time needed to cover the full circle is

$$T = \frac{2\pi}{\omega} \tag{1.22}$$

The distance H that corresponds to this time ($t = T$) is called lead. From the third equation of the set 1.20 one can see that

$$H = bT = \frac{2\pi b}{\omega} \tag{1.23}$$

The tangential unit vector coincides the vector of the absolute velocity $\mathbf{v}$.

$$\mathbf{t}_1 = \frac{\mathbf{v}}{v} \tag{1.24}$$

The absolute velocity $\mathbf{v}$ can be obtained by differentiation of the absolute position vector $\mathbf{r}$ given by the equation 1.19

$$\mathbf{v} = \dot{\mathbf{r}} = \mathbf{I}(-a\omega\sin\omega t) + \mathbf{J}(a\omega\cos\omega t) + \mathbf{K}(b) \tag{1.25}$$

Its length is

$$v = |\mathbf{v}| = \sqrt{(-a\omega\sin\omega t)^2 + (a\omega\cos\omega t)^2 + b^2} = \sqrt{a^2\omega^2 + b^2} \tag{1.26}$$

Introducing equation 1.25 and 1.26 into equation 1.24 one can get

$$\mathbf{t}_1 = \frac{1}{\sqrt{a^2\omega^2 + b^2}}\left(\mathbf{I}(-a\omega\sin\omega t) + \mathbf{J}(a\omega\cos\omega t) + \mathbf{K}(b)\right) \tag{1.27}$$

By definition the normal vector is

$$\mathbf{n}_1 = \frac{\frac{d\mathbf{t}_1}{dt}}{\left|\frac{d\mathbf{t}_1}{dt}\right|} = \tag{1.28}$$

Differentiating the vector 1.27 with respect to time we have

$$\frac{d\mathbf{t}_1}{dt} = \frac{1}{\sqrt{a^2\omega^2 + b^2}}\Big(\mathbf{I}(-a\omega^2\cos\omega t) + \mathbf{J}(-a\omega^2\sin\omega t) + \mathbf{K}(0)\Big) \tag{1.29}$$

Hence

$$\left|\frac{d\mathbf{t}_1}{dt}\right| = \sqrt{\frac{1}{a^2\omega^2 + b^2}\left((-a\omega^2\cos\omega t)^2 + (-a\omega^2\sin\omega t)^2\right) =} \frac{a\omega^2}{\sqrt{a^2\omega^2 + b^2}} \tag{1.30}$$

Introduction of 1.29 and 1.30 into 1.28 yields

$$\begin{aligned}\mathbf{n}_1 &= \frac{1}{a\omega^2}\Big(\mathbf{I}(-a\omega^2\cos\omega t) + \mathbf{J}(-a\omega^2\sin\omega t) + \mathbf{K}(0)\Big) \\ &= \mathbf{I}(-\cos\omega t) + \mathbf{J}(-\sin\omega t) + \mathbf{K}(0)\end{aligned} \tag{1.31}$$

The binormal vector as the vector product of $\mathbf{t}_1$ and $\mathbf{n}_1$ is

$$\begin{aligned}\mathbf{b}_1 &= \mathbf{t}_1 \times \mathbf{n}_1 = \frac{1}{\sqrt{a^2\omega^2 + b^2}}\begin{vmatrix}\mathbf{I} & \mathbf{J} & \mathbf{K} \\ -a\omega\sin\omega t & a\omega\cos\omega t & b \\ -\cos\omega t & -\sin\omega t & 0\end{vmatrix} \\ &= \frac{1}{\sqrt{a^2\omega^2 + b^2}}\left(\mathbf{I}(b\sin\omega t) + \mathbf{J}(-b\cos\omega t) + \mathbf{K}(a\omega)\right)\end{aligned} \tag{1.32}$$

The geometrical interpretation of these unit vectors is given in Fig. 7.

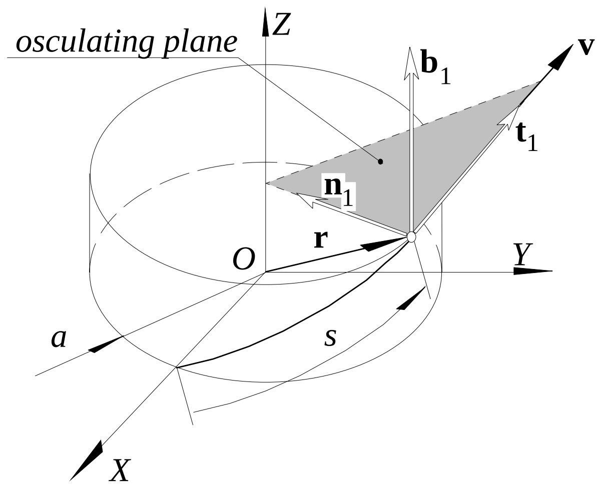


Figure 7

Differentiation of the vector of the absolute velocity yields the absolute acceleration.

$$\mathbf{a} = \dot{\mathbf{v}} = \mathbf{I}\dot{v}_X + \mathbf{J}\dot{v}_Y + \mathbf{K}\dot{v}_Z = \mathbf{I}(-a\omega^2 \cos\omega t) + \mathbf{J}(-a\omega^2 \sin\omega t) + \mathbf{K}(0) \tag{1.33}$$

Its components along the intrinsic system of coordinates are

$$\begin{aligned} a_{\mathbf{t}_1} &= \mathbf{t}_1 \cdot \mathbf{a} \\ a_{\mathbf{n}_1} &= \mathbf{n}_1 \cdot \mathbf{a} \\ a_{\mathbf{b}_1} &= \mathbf{b}_1 \cdot \mathbf{a} \end{aligned} \tag{1.34}$$

Problem 2

The parabolic slide *1* (see Fig. 8) is motionless with respect to the inertial space. The pin *3* follows this parabolic slide and it is driven by the slide *2*. The slide *2* moves with the constant velocity w. Produce

1. the expression for the absolute velocity of the pin *3* as a function of time
2. the expression for the normal and tangential component of the absolute acceleration of the pin as a function of time.

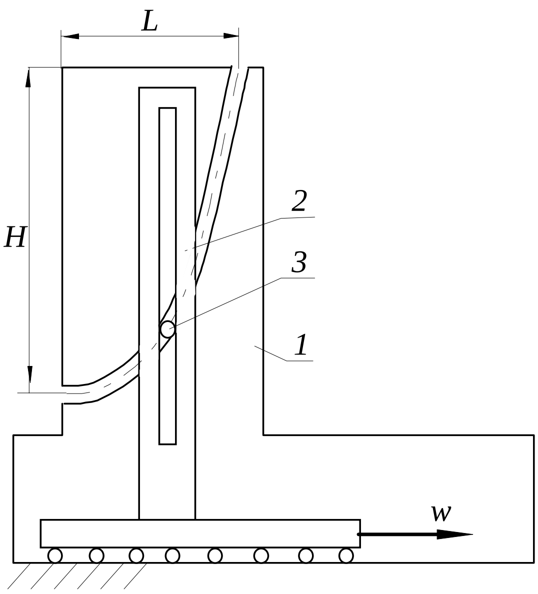


Figure 8

Solution

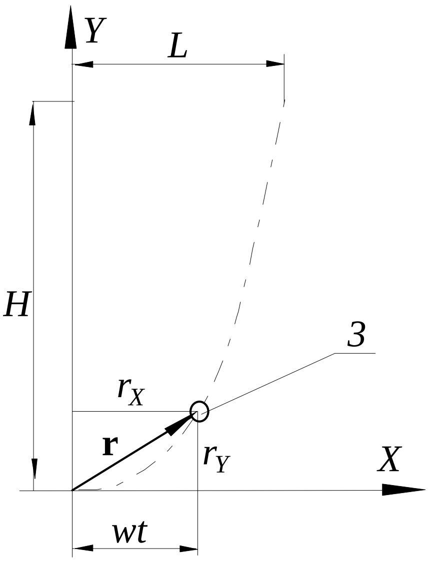


Figure 9

The equation of the parabolic slide is

$$Y = bX^2 \tag{1.35}$$

Since this parabola goes through point (L, H)

$$H = bL^2 \tag{1.36}$$

Hence

$$b = \frac{H}{L^2} \tag{1.37}$$

The displacement along the X coordinate is wt. Therefore the components of the position vector $\mathbf{r}$ (see Fig. 9) are

$$\mathbf{r} = \mathbf{I}r_X + \mathbf{J}r_Y = \mathbf{I}wt + \mathbf{J}bw^2t^2 \tag{1.38}$$

The absolute velocity of the pin, as the first derivative of the absolute position vector with respect to time, is

$$\mathbf{v} = \dot{\mathbf{r}} = \mathbf{I}w + \mathbf{J}2bw^2t \tag{1.39}$$

Its magnitude is

$$|\mathbf{v}| = v = \sqrt{w^2 + 4b^2w^4t^2} = w\sqrt{1 + 4b^2w^2t^2} \tag{1.40}$$

The tangential unit vector is

$$\mathbf{t}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1 + 4b^2w^2t^2}}\,(\mathbf{I} + \mathbf{J}2bwt) \tag{1.41}$$

The normal vector is equal to

$$\mathbf{n}_1 = \frac{\frac{d\mathbf{t}_1}{dt}}{\left|\frac{d\mathbf{t}_1}{dt}\right|} \tag{1.42}$$

where

$$\frac{d\mathbf{t}_1}{dt} = \frac{1}{(1+4b^2w^2t^2)^{\frac{3}{2}}}\left(\mathbf{I}\left(-4b^2w^2t\right) + \mathbf{J}\left(2bw\right)\right) \tag{1.43}$$

and

$$\left|\frac{d\mathbf{t}_1}{dt}\right| = \frac{2bw}{1+4b^2w^2t^2} \tag{1.44}$$

Hence

$$\mathbf{n}_1 = \frac{1}{\sqrt{1+4b^2w^2t^2}}\left(\mathbf{I}\left(-2bwt\right) + \mathbf{J}\left(1\right)\right) \tag{1.45}$$

The absolute acceleration according to 1.39is

$$\mathbf{a} = \dot{\mathbf{v}} = \mathbf{I}(0) + \mathbf{J}2bw^2 \tag{1.46}$$

Its tangential component is

$$\begin{aligned} a_{\mathbf{t}_1} &= \mathbf{t}_1 \cdot \mathbf{a} = \left(\frac{1}{\sqrt{1+4b^2w^2t^2}}\left(\mathbf{I} + \mathbf{J}2bwt\right)\right) \cdot \left(\mathbf{I}(0) + \mathbf{J}2bw^2\right) = \\ &= \frac{4b^2w^3t}{\sqrt{1+4b^2w^2t^2}} \end{aligned} \tag{1.47}$$

The normal component as the dot product of the normal vector $\mathbf{n}_1$ and the acceleration $\mathbf{a}$ is

$$\begin{aligned} a_{\mathbf{n}_1} &= \mathbf{n}_1 \cdot \mathbf{a} = \left(\frac{1}{\sqrt{1+4b^2w^2t^2}}\left(\mathbf{I}\left(-2bwt\right) + \mathbf{J}\left(1\right)\right)\right) \cdot \left(\mathbf{I}(0) + \mathbf{J}2bw^2\right) = \\ &= \frac{2bw^2}{\sqrt{1+4b^2w^2t^2}} \end{aligned} \tag{1.48}$$

Problem 3

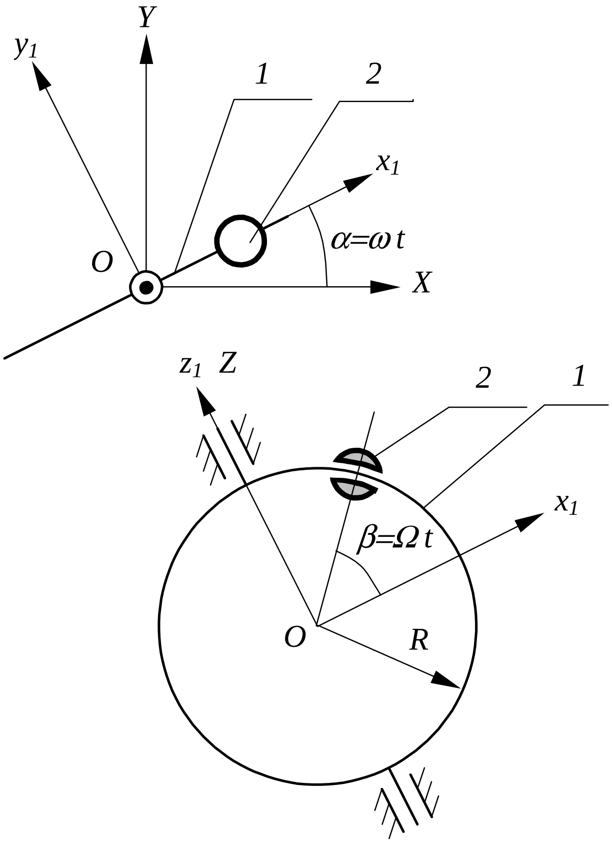


Figure 10

The circular slide 1 of radius R (see Fig. 10) rotates about the vertical axis Z of the inertial system of coordinates XYZ. Its rotation is determined by the angle $\alpha = \omega t$. The system of coordinates $x_1y_1z_1$ is attached to the slide 1. The element 2 can be treated as a particle. The relative motion of the particle 2 with respect to the slide is determined by the angular displacement $\beta = \Omega t$.

Produce:

1. the components of the absolute linear velocity of the particle 2 along the system of coordinates XYZ and $x_1y_1z_1$

2. the components of the absolute linear acceleration of the particle 2 along the system of coordinates XYZ and $x_1y_1z_1$

Solution

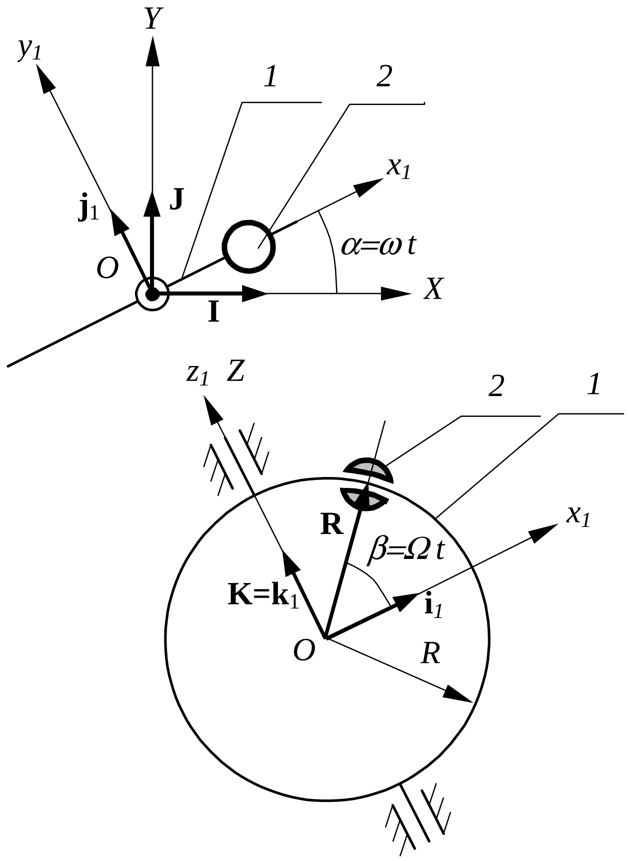


Figure 11

According to Fig. 11, the absolute position vector of the particle is

$$\mathbf{R} = \mathbf{i}_1 R \cos \Omega t + \mathbf{k}_1 R \sin \Omega t \tag{1.49}$$

The absolute velocity of the particle is the time derivative of this vector. To differentiate it one has to produce components of this vector along the inertial system of coordinates.

$$\begin{aligned} R_X &= \mathbf{I} \cdot \mathbf{R} = \mathbf{I} \cdot \mathbf{i}_1 R \cos \Omega t + \mathbf{I} \cdot \mathbf{k}_1 R \sin \Omega t = R \cos \omega t \cos \Omega t \\ R_Y &= \mathbf{J} \cdot \mathbf{R} = \mathbf{J} \cdot \mathbf{i}_1 R \cos \Omega t + \mathbf{J} \cdot \mathbf{k}_1 R \sin \Omega t = R \sin \omega t \cos \Omega t \\ R_Z &= \mathbf{K} \cdot \mathbf{R} = \mathbf{K} \cdot \mathbf{i}_1 R \cos \Omega t + \mathbf{K} \cdot \mathbf{k}_1 R \sin \Omega t = R \sin \Omega t \end{aligned} \tag{1.50}$$

Hence the components of the absolute velocity along the inertial system of coordinates are

$$\begin{aligned} v_X &= -R\omega \sin \omega t \cos \Omega t - R\Omega \cos \omega t \sin \Omega t \\ v_Y &= R\omega \cos \omega t \cos \Omega t - R\Omega \sin \omega t \sin \Omega t \\ v_Z &= R\Omega \cos \Omega t \end{aligned} \tag{1.51}$$

The second derivative yields the components of the absolute acceleration along the inertial system of coordinates XYZ

$$\begin{aligned}
a_X &= -R\omega^2 \cos\omega t \cos\Omega t + 2R\omega\Omega \sin\omega t \sin\Omega t - R\Omega^2 \cos\omega t \cos\Omega t \\
a_Y &= -R\omega^2 \sin\omega t \cos\Omega t - 2R\omega\Omega \cos\omega t \sin\Omega t - R\Omega^2 \sin\omega t \cos\Omega t \\
a_Z &= -R\Omega^2 \sin\Omega t
\end{aligned} \tag{1.52}$$

Now, the components of the absolute velocity and the components of the absolute acceleration can be transfer from the inertial system of coordinates XYZ to the $x_1y_1z_1 system$of coordinates.

$$\begin{aligned}
v_{x1} &= \mathbf{i}_1 \cdot \mathbf{v} = \\
&= \mathbf{i}_1 \cdot \mathbf{I}\,(-R\omega \sin\omega t \cos\Omega t - R\Omega \cos\omega t \sin\Omega t) \\
&\quad + \mathbf{i}_1 \cdot \mathbf{J}\,(R\omega \cos\omega t \cos\Omega t - R\Omega \sin\omega t \sin\Omega t) \\
&\quad + \mathbf{i}_1 \cdot \mathbf{K}\,(R\Omega \cos\Omega t) \\
&= \cos\omega t\,(-R\omega \sin\omega t \cos\Omega t - R\Omega \cos\omega t \sin\Omega t) \\
&\quad + \sin\omega t\,(R\omega \cos\omega t \cos\Omega t - R\Omega \sin\omega t \sin\Omega t) \\
&\quad + 0 \cdot (R\Omega \cos\Omega t) \\
&= -R\Omega \sin\Omega t \\
v_{y1} &= \mathbf{j}_1 \cdot \mathbf{v} = R\omega \cos\Omega t \\
v_{z1} &= \mathbf{k}_1 \cdot \mathbf{v} = R\Omega \cos\Omega t
\end{aligned} \tag{1.53}$$

Similarly

$$\begin{aligned}
a_{x1} &= \mathbf{i}_1 \cdot \mathbf{a} = -R\Omega^2 \cos\Omega t - R\omega^2 \cos\Omega t \\
a_{y1} &= \mathbf{j}_1 \cdot \mathbf{a} = -2R\omega\Omega \sin\Omega t \\
a_{z1} &= \mathbf{k}_1 \cdot \mathbf{a} = -R\Omega^2 \sin\Omega t
\end{aligned} \tag{1.54}$$

Problem 4

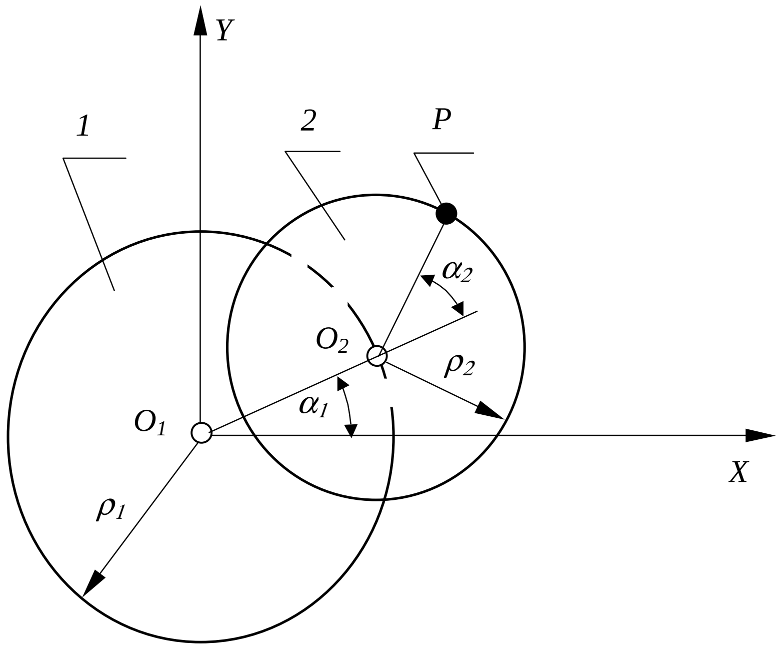


Figure 12

Figure 12 shows the sketch of a Ferris wheel. Its base 1, of radius ρ_1, rotates about the horizontal axis Z of the inertial system of coordinates XYZ. Its instantaneous position is determined by the angular displacement α_1. The wheel 2, of radius ρ_2, rotates about axis parallel to Z through the point O_2. Its relative rotation is determined by the angular position α_2.

Produce

1. the equation of trajectory of the point P

Answer:

$$\begin{cases} X = \rho_1 \cos\alpha_1 + \rho_2 \cos(\alpha_1 + \alpha_2) \\ Y = \rho_1 \sin\alpha_1 + \rho_2 \sin(\alpha_1 + \alpha_2) \end{cases}$$

2. the expression for the components of the absolute velocity of the point O_2

Answer:

$v_{O_2X} = -\rho_1\dot{\alpha}_1 \sin\alpha_1$

$v_{O_2Y} = +\rho_1\dot{\alpha}_1 \cos\alpha_1$

3. the expression for the components of the absolute acceleration of the point O_2

Answer:

$a_{O_2X} = -\rho_1\ddot{\alpha}_1 \sin\alpha_1 - \rho_1\dot{\alpha}_1^2 \cos\alpha_1$

$a_{O_2Y} = +\rho_1\ddot{\alpha}_1 \cos\alpha_1 - \rho_1\dot{\alpha}_1^2 \sin\alpha_1$

4. the expression for the components of the absolute velocity of the point P

Answer:

$v_{PX} = -\rho_1\dot{\alpha}_1 \sin\alpha_1 - \rho_2(\dot{\alpha}_1 + \dot{\alpha}_2)\sin(\alpha_1 + \alpha_2)$

$v_{PY} = +\rho_1\dot{\alpha}_1 \cos\alpha_1 + \rho_2(\dot{\alpha}_1 + \dot{\alpha}_2)\cos(\alpha_1 + \alpha_2)$

5. the expression for the components of the absolute acceleration of the point P

Answer:
$a_{PX} = -\rho_1\ddot{\alpha}_1 \sin\alpha_1 - \rho_1\dot{\alpha}_1^2 \cos\alpha_1 - \rho_2(\ddot{\alpha}_1 + \ddot{\alpha}_2)\sin(\alpha_1+\alpha_2) - \rho_2(\dot{\alpha}_1+\dot{\alpha}_2)^2 \cos(\alpha_1+\alpha_2)$
$a_{PY} = +\rho_1\ddot{\alpha}_1 \cos\alpha_1 - \rho_1\dot{\alpha}_1^2 \sin\alpha_1 + \rho_2(\ddot{\alpha}_1 + \ddot{\alpha}_2)\cos(\alpha_1+\alpha_2) - \rho_2(\dot{\alpha}_1+\dot{\alpha}_2)^2 \sin(\alpha_1+\alpha_2)$

Problem 5

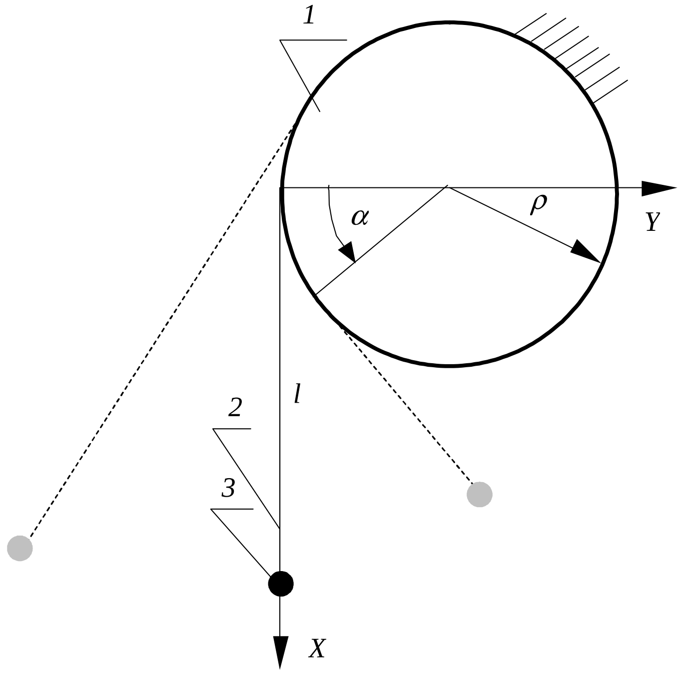


Figure 13

The particle 3 and the rope 2 of length l (see Fig. 37) forms a pendulum operating in the vertical plane XY of an inertial space. It is suspended from the cylinders 1 of radius ρ The instantaneous position of the pendulum can be determined by the angular displacement α.

Produce

1. the expression for the components of the absolute velocity of the particle 3 along the inertial system of coordinates XYZ as a function of the angular displacement α

Answer:

$v_X = -l\dot{\alpha}\sin\alpha + \varrho\alpha\dot{\alpha}\sin\alpha$

$v_Y = +l\dot{\alpha}\cos\alpha - \varrho\alpha\dot{\alpha}\cos\alpha$

2. the expression for the components of the absolute acceleration of the particle 3 along the inertial system of coordinates XYZ as a function of the angular displacement α

Answer:

$a_X = -l\ddot{\alpha}\sin\alpha - l\dot{\alpha}^2\cos\alpha + \varrho\dot{\alpha}^2\sin\alpha + \varrho\alpha\ddot{\alpha}\sin\alpha + \varrho\alpha\dot{\alpha}^2\cos\alpha$

$a_Y = +l\ddot{\alpha}\cos\alpha - l\dot{\alpha}^2\sin\alpha - \varrho\dot{\alpha}^2\cos\alpha - \varrho\alpha\ddot{\alpha}\cos\alpha + \varrho\alpha\dot{\alpha}^2\sin\alpha$

1.2 KINETICS OF PARTICLE

1.2.1 *Equations of motion*

Newton second law for a single particle can be formulated in the following form

$$\mathbf{F} = \frac{d}{dt}(m\dot{\mathbf{r}}) \tag{1.55}$$

where

$\mathbf{F}$ - is the resultant force acting on the particle
m - is mass of the particle
$\dot{\mathbf{r}}$ - is the absolute velocity.

If the mass is constant, the second law could be simplify to the following form

$$\mathbf{F} = m\ddot{\mathbf{r}} \tag{1.56}$$

where $\ddot{\mathbf{r}}$ is called acceleration of the particle.

If the position vector $\mathbf{r}$ and force $\mathbf{F}$ are determined by their components along the same system of coordinates (eg $\mathbf{r} = \mathbf{I}r_X + \mathbf{J}r_Y + \mathbf{K}r_Z$, $F = \mathbf{I}F_X + \mathbf{J}F_Y + \mathbf{K}F_Z$) the above equation is equivalent to three scalar equations.

$$\begin{aligned} F_X &= m\ddot{r}_X \\ F_Y &= m\ddot{r}_Y \\ F_Z &= m\ddot{r}_Z \end{aligned} \tag{1.57}$$

If number of particles n is relatively low, we are able to produce a Free Body Diagram for each of these particle separately and create *3n* differential equations that permit each dynamic problem to be solved.

1.2.2 *Work*

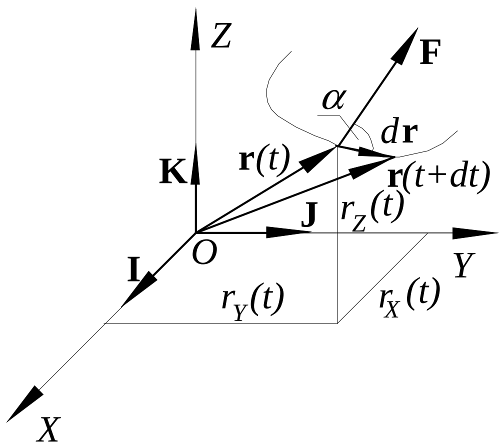


Figure 14

Let us assume that the point of application of a force $\mathbf{F}$ slides along the path determined by the position vector $\mathbf{r}$ (see Fig.14).

DEFINITION: It is said that the work done by the force F on the displacement $d\mathbf{r}$ is equal to the dot product of these two vectors.

$$dW = \mathbf{F}\cdot d\mathbf{r} \tag{1.58}$$

If both, the force and the position vector are given by their components along the inertial system of coordinates XYZ

$$\mathbf{F} = \mathbf{I}F_X(t) + \mathbf{J}F_Y(t) + \mathbf{K}F_Z(t) \qquad \mathbf{r} = \mathbf{I}r_X(t) + \mathbf{J}r_Y(t) + \mathbf{K}r_Z(t) \tag{1.59}$$

since $d\mathbf{r}$ is

$$d\mathbf{r} = \mathbf{v}dt = (\mathbf{I}\dot{r}_X(t) + \mathbf{J}\dot{r}_Y(t) + \mathbf{K}\dot{r}_Z(t))\,dt \tag{1.60}$$

the work, according to the above definition, is

$$\begin{aligned} dW &= (\mathbf{I}F_X(t) + \mathbf{J}F_Y(t) + \mathbf{K}F_Z(t)) \cdot (\mathbf{I}\dot{r}_X(t) + \mathbf{J}\dot{r}_Y(t) + \mathbf{K}\dot{r}_Z(t))\,dt \\ &= F_X(t)\dot{r}_X(t) + F_Y(t)\dot{r}_Y(t) + F_Z(t)\dot{r}_Z(t)dt \end{aligned} \tag{1.61}$$

Hence, the work done between the instant t_1 and t_2 is

$$W = \int_{t_1}^{t_2} (F_X(t)\dot{r}_X(t) + F_Y(t)\dot{r}_Y(t) + F_Z(t)\dot{r}_Z(t))\,dt \tag{1.62}$$

If the force $\mathbf{F}$ is given as a function of the displacement s (see Fig. 15), we have

$$dW = \mathbf{F}\cdot d\mathbf{r} = F\cos\alpha ds = F_t ds \tag{1.63}$$

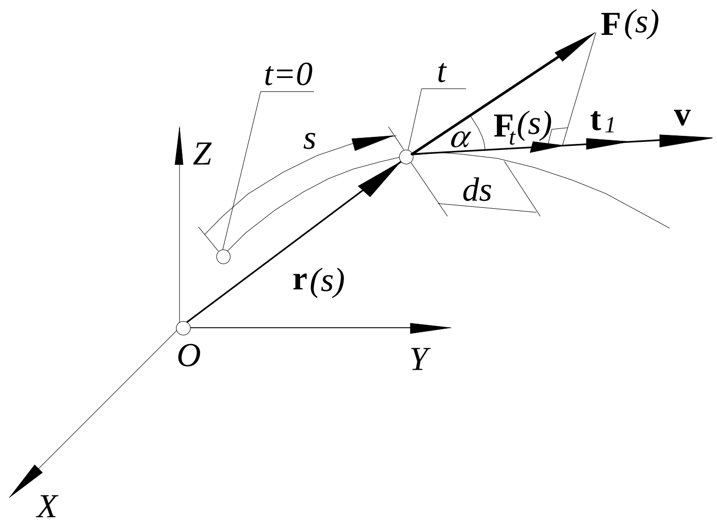


Figure 15

Hence, the total work done by the force between the position s_1 and s_2 is

$$W = \int_{s_1}^{s_2} F_t ds \tag{1.64}$$

1.2.3 Kinetic energy

DEFINITION: The scalar magnitude determined by the following expression

$$T = \frac{1}{2} m \mathbf{v}^2 \tag{1.65}$$

where m is mass of a particle and v is its absolute velocity, is called the *kinetic energy* of the particle.

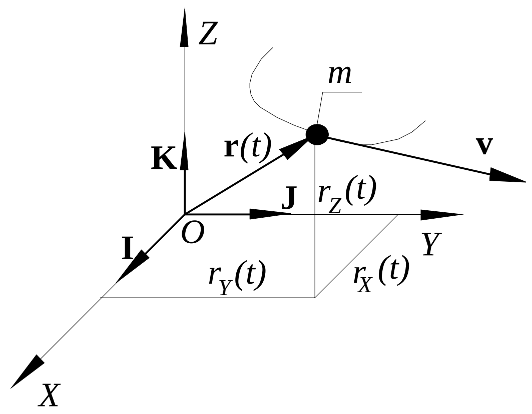


Figure 16

If motion of the particle is given by means of the absolute position vector $\mathbf{r}$ (see Fig. 16) its velocity is

$$\mathbf{v} = \mathbf{I}\dot{r}_X(t) + \mathbf{J}\dot{r}_Y(t) + \mathbf{K}\dot{r}_Z(t) \tag{1.66}$$

The kinetic energy is

$$T = \frac{1}{2} m \mathbf{v}^2 = \frac{1}{2} m \left((\dot{r}_X(t))^2 + (\dot{r}_Y(t))^2 + (\dot{r}_Z(t))^2 \right) \tag{1.67}$$

Principle of work and energy

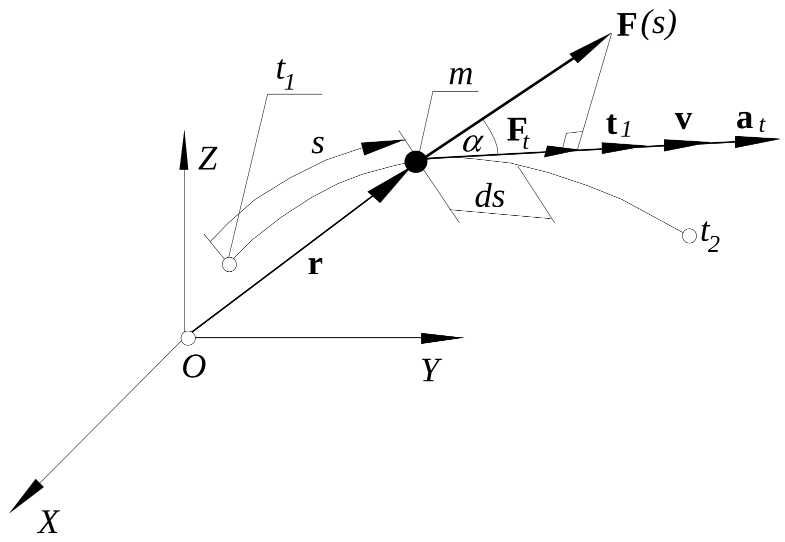


Figure 17

Let **F** (see Fig. 17) be the resultant of all forces acting on the particle of mass m. According to the formula 1.63 we have

$$dW = F_t ds \tag{1.68}$$

Utilization of the second Newton's law yields the following expression for the work

$$dW = m a_t ds \tag{1.69}$$

But the absolute tagential acceleration is

$$a_t = \frac{dv}{dt} = \frac{dv}{dt}\frac{ds}{ds} = \frac{dv}{dt}\frac{vdt}{ds} = \frac{vdv}{ds} \tag{1.70}$$

Introduction of expression 1.70 into 1.69 results in the following formula for the work dW

$$dW = mvdv \tag{1.71}$$

Integration of the above equation within the range of time between the instant t_1 and t_2 yields

$$W = \int_{v_1}^{v_2} mvdv = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = T_2 - T_1 \tag{1.72}$$

The last relationship is known as the *work and energy principle.*

STATEMENT: The increment in the kinetic energy of a particle is equal to the work done by the resultant of all forces acting on the particle.

1.2.4 ***Power***

DEFINITION: The rate of change of the work performed by a force is called *power* **generated by this force.**

$$P = \frac{dW}{dt} \tag{1.73}$$

Since according to equation 1.58

$$dW = \mathbf{F} \cdot d\mathbf{r} \tag{1.74}$$

the power is

$$P = \frac{\mathbf{F} \cdot d\mathbf{r}}{dt} = \mathbf{F} \cdot \mathbf{v} \tag{1.75}$$

1.2.5 ***Problems***

Problem 6

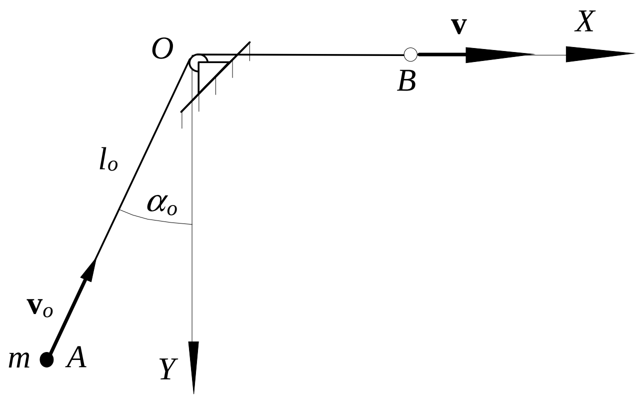


Figure 18

An element that can be considered as a particle of mass m (see Fig. 18) is suspended on the massless rope AOB in the vertical plane of the inertial space XYZ. The end B of this rope moves with the constant velocity $\mathbf{v}$. When the length AO of the rope was equal to l_o and its angular position was equal to α_o, the particle was freed to move with the initial velocity $v_0 = v$. Produce the differential equation of motion of the particle. Solve numerically the equations of motion and produce trajectory of motion of the particle for the following set of the numerical data. $m = 1kg$, $l_o = 1m$, $\alpha_o = 1rad$, $v = 0.1m/s$

Solution

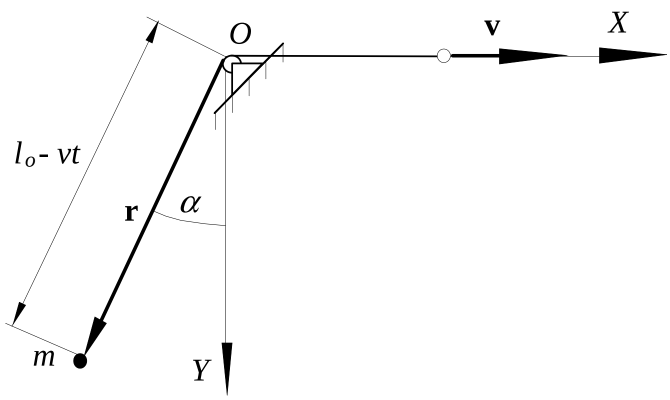


Figure 19

The instantaneous position of the particle is determined by the absolute position vector $\mathbf{r}$. Its components, according to Fig. 19, along the inertial system of coordinates are

$$\mathbf{r} = \mathbf{I}(-(l_0 - vt)\sin\alpha) + \mathbf{J}((l_0 - vt)\cos\alpha) \tag{1.76}$$

Differentiation of the position vector with respect to time yields vector of the absolute velocity and the absolute acceleration of the particle.

$$\dot{\mathbf{r}} = \mathbf{I}(-l_0\dot{\alpha}\cos\alpha + v\sin\alpha + vt\dot{\alpha}\cos\alpha) + \mathbf{J}(-l_0\dot{\alpha}\sin\alpha - v\cos\alpha + vt\dot{\alpha}\sin\alpha) \tag{1.77}$$

and

$$\ddot{\mathbf{r}} = \mathbf{I}(a_X) + \mathbf{J}(a_Y) \tag{1.78}$$

where

$$\begin{aligned} a_X &= -l_0\ddot{\alpha}\cos\alpha + l_0\dot{\alpha}^2\sin\alpha + 2v\dot{\alpha}\cos\alpha + vt\ddot{\alpha}\cos\alpha - vt\dot{\alpha}^2\sin\alpha \\ a_Y &= -l_0\ddot{\alpha}\sin\alpha - l_0\dot{\alpha}^2\cos\alpha + 2v\dot{\alpha}\sin\alpha + vt\ddot{\alpha}\sin\alpha + vt\dot{\alpha}^2\cos\alpha \end{aligned} \tag{1.79}$$

Fig. 20 presents the free body diagram for the particle considered. Application of the Newton's law to this particle results in two equations with the two unknowns R and α.

$$\begin{aligned} ma_X &= R\sin\alpha \\ ma_Y &= mg - R\cos\alpha \end{aligned} \tag{1.80}$$

The first equation allows the unknown interaction force R to be determined.

$$R = \frac{ma_X}{\sin\alpha} \tag{1.81}$$

Introduction of the above expression into the second equation of 1.80 yields

$$ma_Y = mg - \frac{ma_X}{\sin\alpha}\cos\alpha \tag{1.82}$$

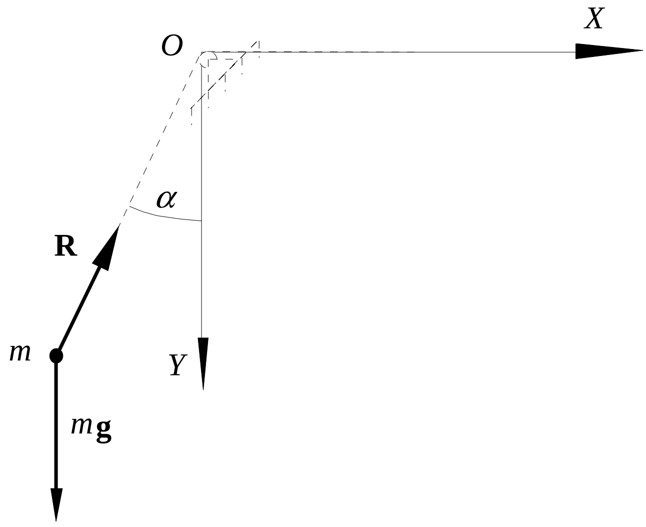


Figure 20

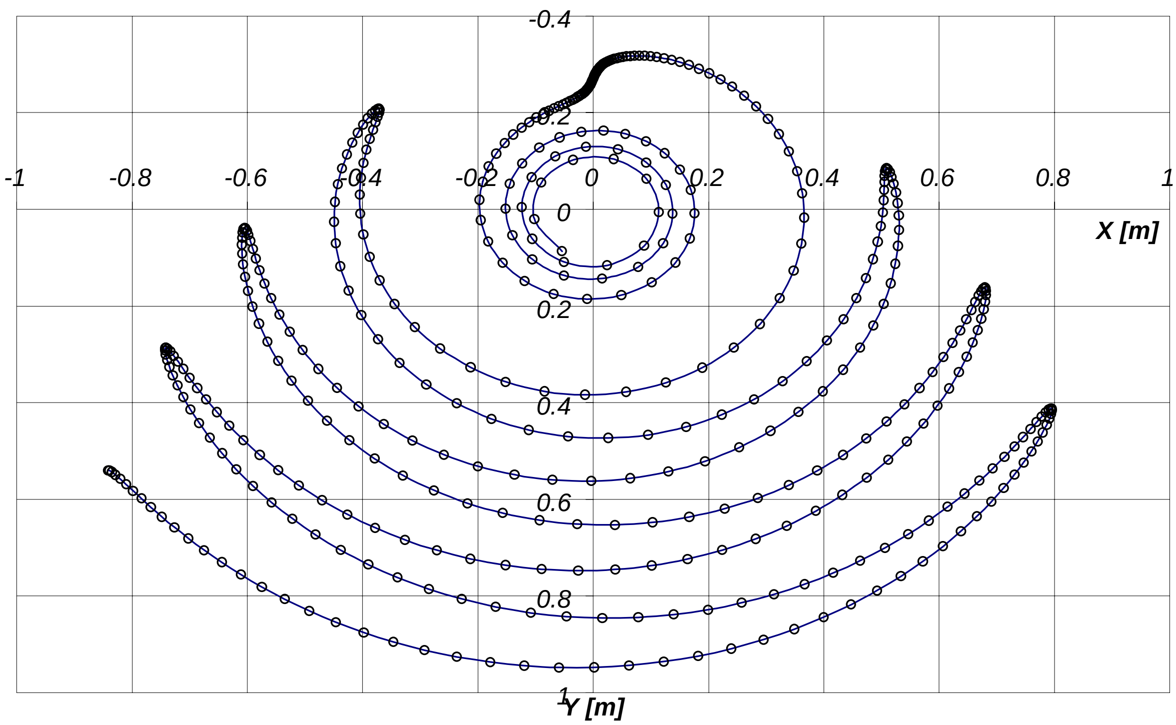


Figure 21

Introduction of the expressions 1.79 into equation 1.82 yields the differential equation of motion that after simplification takes the following form

$$(l_o - vt)\ddot{\alpha} - 2v\dot{\alpha} + g\sin\alpha = 0 \tag{1.83}$$

This equation must fulfil the following initial conditions

$$\alpha|_{t=0} = \alpha_0 \qquad and \qquad \dot{\alpha}|_{t=0} = 0 \tag{1.84}$$

Solution of the above equation as a function of time is presented in Fig. 21.The dots in this diagram represent subsequent positions of the particle after equal intervals of time equal to $0.02s$.

Problem 7

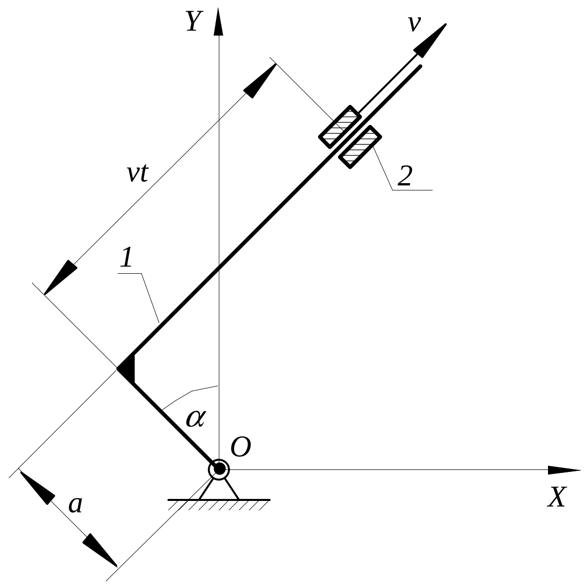


Figure 22

The slide 1 (see Fig. 22) rotates about the horizontal axis Z of the inertial system of coordinates XYZ. Its instantaneous position with respect to the inertial system of coordinates is determined by the angle $\alpha(t)$. The slider 2 that can be treated as a particle of mass m moves along the slide with the constant velocity v.

Produce

1. the expression for the absolute velocity of the particle 2
2. the expression for the absolute acceleration of the particle 2
3. the expression for the interaction force between the particle and the slide
4. the expression for the driving force that must be applied to the particle to move it with the assumed velocity v.
5. the expression for the work produced by the gravity force

Solution

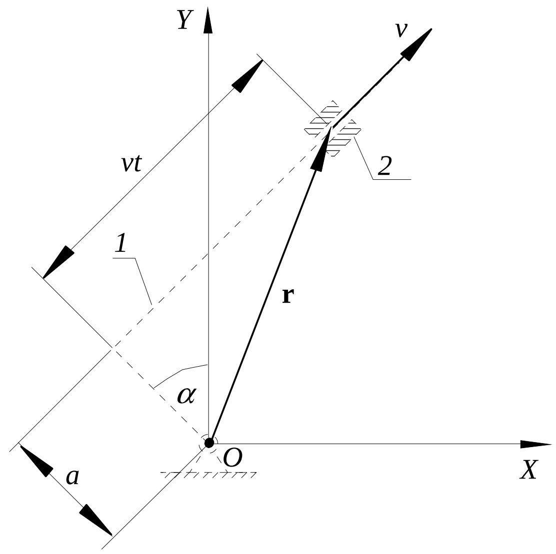


Figure 23

The absolute position vector $\mathbf{r}$, according to the drawing in Fig. 23, is

$$\mathbf{r} = \mathbf{I}(-a\sin\alpha + vt\cos\alpha) + \mathbf{J}(a\cos\alpha + vt\sin\alpha) \tag{1.85}$$

The vector of the absolute velocity can be obtained by differentiation of the absolute position vector with respect to time.

$$\mathbf{v} = \mathbf{I}(-a\dot{\alpha}\cos\alpha + v\cos\alpha - vt\dot{\alpha}\sin\alpha) + \mathbf{J}(-a\dot{\alpha}\sin\alpha + v\sin\alpha + vt\dot{\alpha}\cos\alpha) \tag{1.86}$$

Similarly, the absolute acceleration is the time derivative of the absolute velocity. Hence

$$\mathbf{a} = \mathbf{I}a_X + \mathbf{J}a_Y \tag{1.87}$$

where

$$\begin{aligned} a_X &= -a\ddot{\alpha}\cos\alpha + a\dot{\alpha}^2\sin\alpha - 2v\dot{\alpha}\sin\alpha - vt\ddot{\alpha}\sin\alpha - vt\dot{\alpha}^2\cos\alpha \\ a_Y &= -a\ddot{\alpha}\sin\alpha - a\dot{\alpha}^2\cos\alpha + 2v\dot{\alpha}\cos\alpha + vt\ddot{\alpha}\cos\alpha - vt\dot{\alpha}^2\sin\alpha \end{aligned} \tag{1.88}$$

The interaction force $\mathbf{N}$ and the driving force $\mathbf{D}$ (see the Free Body Diagram in Fig. 24) can be obtained by application of the Newton's equation to the particle.

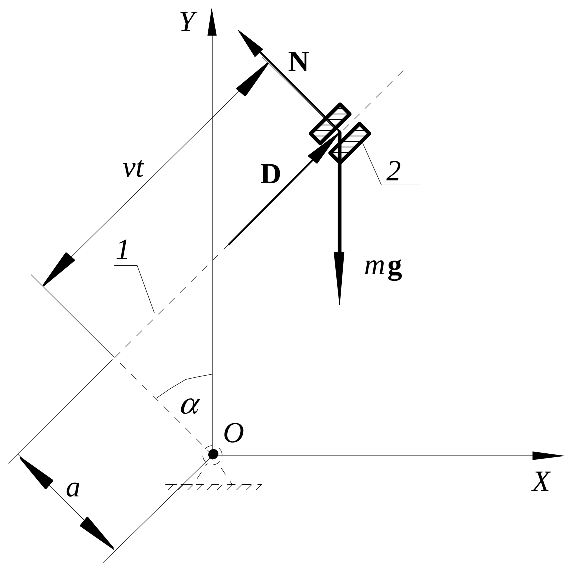


Figure 24

Hence

$$\begin{aligned} ma_X &= D\cos\alpha - N\sin\alpha \\ ma_Y &= D\sin\alpha + N\cos\alpha - mg \end{aligned} \tag{1.89}$$

Solving these equations with respect to the unknown driving force D and the normal interaction force N one can get

$$D = m(a_X\cos\alpha + a_Y\sin\alpha + g\sin\alpha) \tag{1.90}$$

and

$$N = m(a_X\sin\alpha - a_Y\cos\alpha - g\cos\alpha) \tag{1.91}$$

The work produced by the gravity force is

$$\begin{aligned} W &= \int_0^t \mathbf{J}(-mg)\cdot\mathbf{v}d\tau \\ &= \int_0^t \mathbf{J}(-mg)\cdot\mathbf{I}(-a\dot{\alpha}\cos\alpha + v\cos\alpha - v\tau\dot{\alpha}\sin\alpha) + \\ &\qquad\qquad +\mathbf{J}(-a\dot{\alpha}\sin\alpha + v\sin\alpha + v\tau\dot{\alpha}\cos\alpha))d\tau \\ &= \int_0^t (-mg)(-a\dot{\alpha}\sin\alpha + v\sin\alpha + v\tau\dot{\alpha}\cos\alpha))d\tau = \\ &= -mg(a\cos\alpha + v\tau\sin\alpha)\big|_0^t \end{aligned} \tag{1.92}$$

If $\alpha(0) = 0$

$$W = +mg\left(a(1-\cos\alpha) - vt\sin\alpha\right) \tag{1.93}$$

This work, taken with sign -, is equal to the increment of the potential energy of the particle.

Problem 8

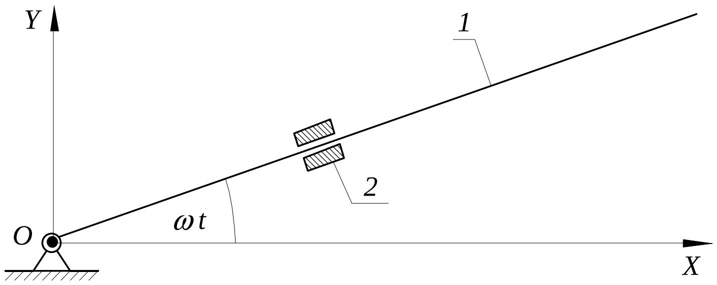


Figure 25

The slide *1* (see Fig. 25) rotates with the constant angular velocity ω with respect to the vertical plane XY of the inertial system of coordinates XYZ. The body *2* that can be treated as a particle of mass m can move along the slide *1* without friction. Produce the equations of motion of the particle.

Solution

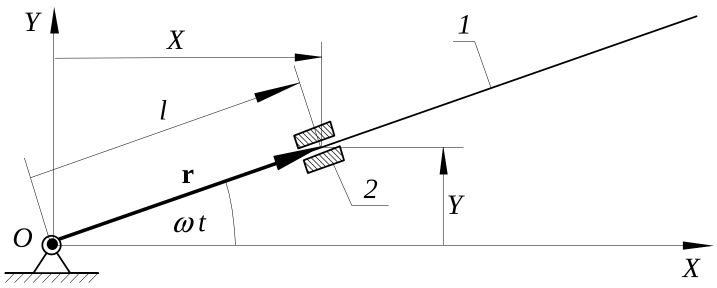


Figure 26

Let l be the instantaneous distance between the particle and the origin of the inertial system of coordinates XY. The components of the absolute position vector of the particle is

$$\mathbf{r} = \mathbf{I}X + \mathbf{J}Y = \mathbf{I}l\cos\omega t + \mathbf{J}l\sin\omega t \tag{1.94}$$

Since l is a function of time its first and second derivative is

$$\begin{aligned}
\dot{\mathbf{r}} &= \mathbf{I}(\dot{l}\cos\omega t - l\omega\sin\omega t) + \mathbf{J}(\dot{l}\sin\omega t + l\omega\cos\omega t) \\
\ddot{\mathbf{r}} &= \mathbf{I}(\ddot{l}\cos\omega t - 2\dot{l}\omega\sin\omega t - l\omega^2\cos\omega t) + \mathbf{J}(\ddot{l}\sin\omega t + 2\dot{l}\omega\cos\omega t - l\omega^2\sin\omega t)
\end{aligned} \tag{1.95}$$

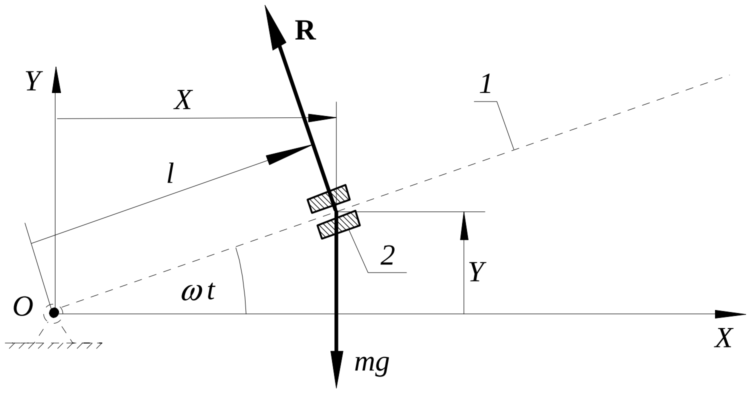


Figure 27

The second derivative represents the absolute acceleration of the particle. Hence, the application of the Newton's equation to the free body diagram shown in Fig. 27 yields.

$$\begin{aligned}
m(\ddot{l}\cos\omega t - 2\dot{l}\omega\sin\omega t - l\omega^2\cos\omega t) &= -R\sin\omega t \\
m(\ddot{l}\sin\omega t + 2\dot{l}\omega\cos\omega t - l\omega^2\sin\omega t) &= R\cos\omega t - mg
\end{aligned} \tag{1.96}$$

Elimination of the unknown reaction R from the above set of equations allows the equation of motion to be produced in the following form.

$$\ddot{l} - l\omega^2 = -g\sin\omega t \tag{1.97}$$

Since the equitation is linear with respect to the distance l, it can be analytically solved. Its solution is

$$l = Ae^{\omega t} + Be^{-\omega t} + \frac{g}{2\omega^2}\sin\omega t \tag{1.98}$$

where A and B are constant magnitudes. For the following initial conditions

$$\begin{aligned} l \ &| \ _{t=0} = 0 \\ \dot{l} \ &| \ _{t=0} = 0 \end{aligned} \tag{1.99}$$

the constants A and B are

$$A = -\frac{g}{4\omega^2}; \qquad B = \frac{g}{4\omega^2} \tag{1.100}$$

and the equation of motion takes form

$$l = -\frac{g}{4\omega^2}e^{\omega t} + \frac{g}{4\omega^2}e^{-\omega t} + \frac{g}{2\omega^2}\sin\omega t \tag{1.101}$$

The parametric equations of the trajectory of the particle are

$$\begin{aligned} X &= \left(-\frac{g}{4\omega^2}e^{\omega t} + \frac{g}{4\omega^2}e^{-\omega t} + \frac{g}{2\omega^2}\sin\omega t\right)\cos\omega t \\ Y &= \left(-\frac{g}{4\omega^2}e^{\omega t} + \frac{g}{4\omega^2}e^{-\omega t} + \frac{g}{2\omega^2}\sin\omega t\right)\sin\omega t \end{aligned} \tag{1.102}$$

It is presented in Fig. 28 for $\omega = 1$

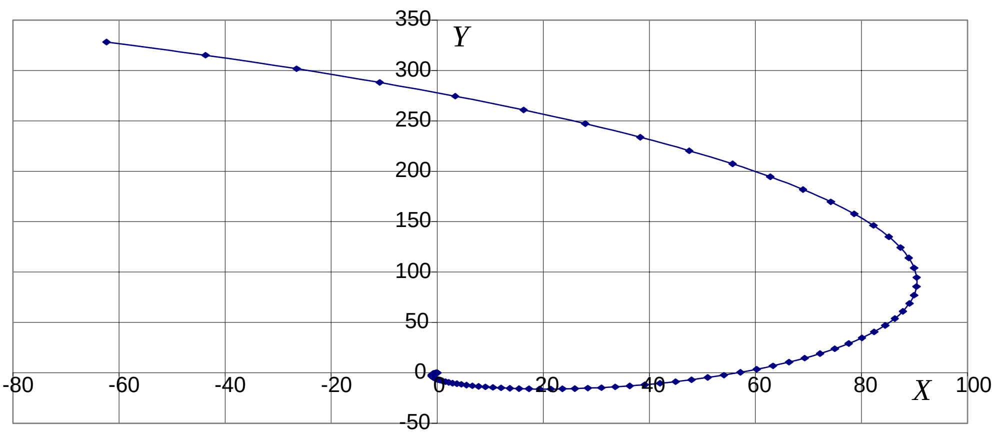


Figure 28

Problem 9

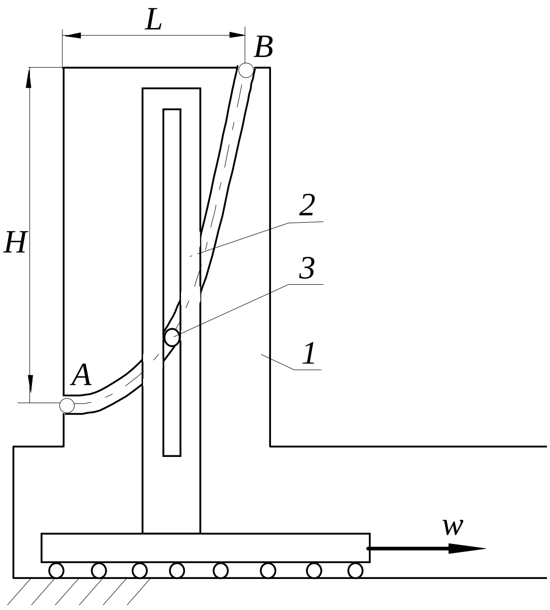


Figure 29

The parabolic slide *1* (see Fig. 29) is motionless with respect to the vertical plane of the inertial space. The pin *3* of mass m follows this parabolic slide and it is driven by the slide *2*. The slide *2* moves with the constant velocity w. The friction between the slides and the pin 3 can be neglected. Produce

1. the expression for the interaction force between the pin 3, and the both slides 1 and 2.
2. the kinetic energy of the particle at the position A and B.
3. the work produced by all the forces acting on the particle
4. verify your results by means of the work-energy principle

Solution

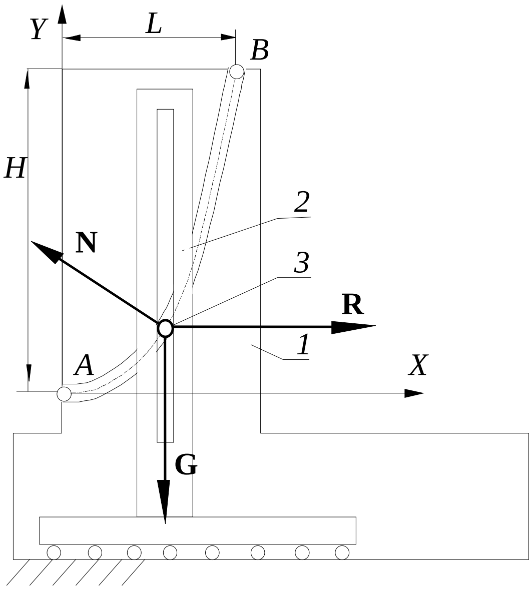


Figure 30

Let us take advantage of the second Newton's law

$$m\mathbf{a} = \mathbf{F} \tag{1.103}$$

The absolute acceleration **a** according to the equation 1.46 is

$$\mathbf{a} = \mathbf{I}(0) + \mathbf{J}(2bw^2) \tag{1.104}$$

The resultant **F** of all forces acting on the particle, according to the Free Body Diagram shown in Fig. 30 is

$$\mathbf{F} = \mathbf{G} + \mathbf{N} + \mathbf{R} \tag{1.105}$$

where

$$\mathbf{G} = \mathbf{J}(-mg) \tag{1.106}$$

$$\mathbf{N} = \mathbf{n}_1 N \tag{1.107}$$

$$\mathbf{R} = \mathbf{I}R \tag{1.108}$$

According to 1.45

$$\mathbf{n}_1 = \frac{1}{\sqrt{1+4b^2w^2t^2}}\left(\mathbf{I}\left(-2bwt\right) + \mathbf{J}\left(1\right)\right) \tag{1.109}$$

Hence the equation 2.47 can be rewritten in the following form

$$\mathbf{J}2bw^2m = \mathbf{J}\left(-mg\right) + \frac{N}{\sqrt{1+4b^2w^2t^2}}\left(\mathbf{I}\left(-2bwt\right) + \mathbf{J}\left(1\right)\right) + \mathbf{I}R \tag{1.110}$$

This vector equation is equivalent to two scalar equations

$$\begin{aligned} 0 &= \frac{-2bwtN}{\sqrt{1+4b^2w^2t^2}} + R \\ 2bw^2m &= -mg + \frac{N}{\sqrt{1+4b^2w^2t^2}} \end{aligned} \tag{1.111}$$

Solving these equations with respect to the unknown N and R we have

$$\begin{aligned} N &= \sqrt{1+4b^2w^2t^2}m(2bw^2+g) \\ R &= 2bwm(2bw^2+g)t \end{aligned} \tag{1.112}$$

The kinetic energy of the particle is

$$T = \frac{1}{2}m(v(t))^2 \tag{1.113}$$

The time associated with the position A is $t_A = 0$. Hence the kinetic energy at the position A according to 1.40 is

$$T_A = \frac{1}{2}m\left(w\sqrt{1+4b^2w^2t_A^2}\right)^2 = \frac{1}{2}mw^2 \tag{1.114}$$

The time associated with the position B is $t_B = \frac{L}{w}$. Hence the kinetic energy at the position B is

$$T_B = \frac{1}{2}m\left(w\sqrt{1+4b^2w^2t_B^2}\right)^2 = \frac{1}{2}mw^2\left(1+4b^2L^2\right) \tag{1.115}$$

Since the normal force $\mathbf{N}$ is perpendicular to the trajectory of the particle, the work done by this force is equal to zero.

$$W_N = 0 \tag{1.116}$$

The work produced by the force $\mathbf{R}$ is

$$W_R = \int_{t_A}^{t_B} \mathbf{R}\cdot d\mathbf{r} = \int_{t_A}^{t_B} \mathbf{R}\cdot\mathbf{v}dt \tag{1.117}$$

According to 1.39

$$\mathbf{v} = \dot{\mathbf{r}} = \mathbf{I}w + \mathbf{J}2bw^2t \tag{1.118}$$

Hence

$$\begin{aligned} W_R &= \int_{t_A}^{t_B} \mathbf{I}2bwm(2bw^2+g)t\cdot\left(\mathbf{I}w + \mathbf{J}2bw^2t\right)dt = \int_{t_A}^{t_B} 2bw^2m(2bw^2+g)tdt \\ &= bmL^2(2bw^2+g) \end{aligned} \tag{1.119}$$

Similarly

$$W_G = \int_{t_A}^{t_B} \mathbf{G} \cdot d\mathbf{r} = \int_{t_A}^{t_B} \mathbf{J}\,(-mg) \cdot \left(\mathbf{I}w + \mathbf{J}2bw^2t\right) dt = \int_{t_A}^{t_B} \left(-2mgbw^2t\right) dt = -mgbL^2 \tag{1.120}$$

The work-kinetic energy principle says that

$$W_N + W_R + W_G = T_B - T_A \tag{1.121}$$

Indeed

$$0 + bmL^2(2bw^2 + g) + \left(-mgbL^2\right) = \frac{1}{2}mw^2\left(1 + 4b^2L^2\right) - \frac{1}{2}mw^2 \tag{1.122}$$

Problem 10

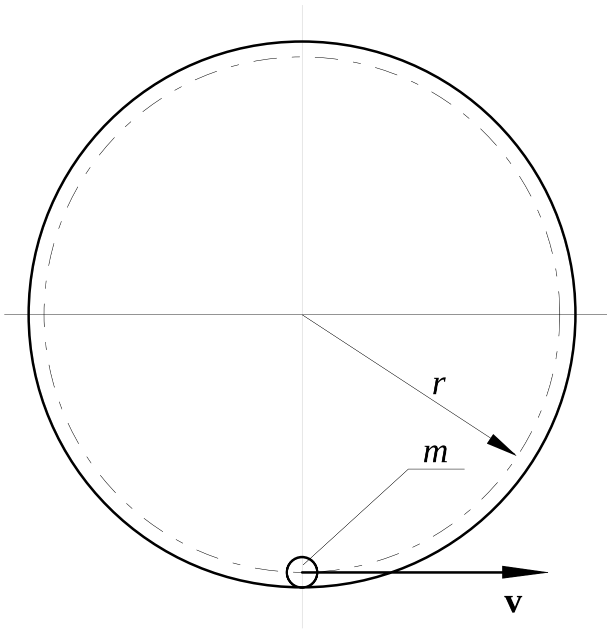


Figure 31

The axis of symmetry the cylinder shown in Fig. 31 coincides the horizontal axis of an inertial system of coordinates. Its radius is equal to r. The initial velocity of the ball of mass m at the position shown in Fig. 31 is v. It rolls over the inner surface of the cylinder. Produce the equation of motion of the ball and the expression for the normal interaction force between the ball and the cylinder.

Solution

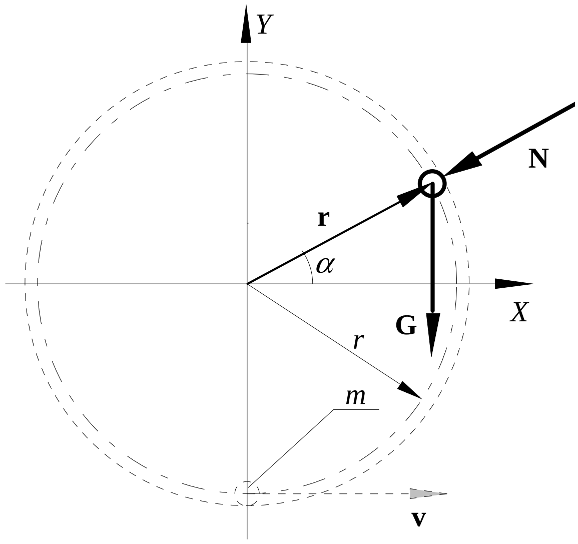


Figure 32

Motion of the ball is determined by the position vector **r**.(see Fig. 32)

$$\mathbf{r} = \mathbf{I}r\cos\alpha + \mathbf{J}r\sin\alpha \tag{1.123}$$

where $\alpha$is an unknown function of time. Two subsequent differentiations yields the absolute acceleration of the ball.

$$\dot{\mathbf{r}} = \mathbf{I}\left(-r\dot{\alpha}\sin\alpha\right) + \mathbf{J}\left(r\dot{\alpha}\cos\alpha\right) \tag{1.124}$$

$$\ddot{\mathbf{r}} = \mathbf{I}\left(-r\ddot{\alpha}\sin\alpha - r\dot{\alpha}^2\cos\alpha\right) + \mathbf{J}\left(r\ddot{\alpha}\cos\alpha - r\dot{\alpha}^2\sin\alpha\right) \tag{1.125}$$

Application of the second Newton's law to the ball yields

$$m\ddot{\mathbf{r}} = \mathbf{G} + \mathbf{N} \tag{1.126}$$

where

$$\mathbf{G} = \mathbf{J}(-mg) \qquad \mathbf{N} = \mathbf{I}(-N\cos\alpha) + \mathbf{J}(-N\sin\alpha) \tag{1.127}$$

Introducing the expressions 1.125 and 1.127 into the equation 1.126one may get

$$\begin{aligned} & \mathbf{I}\left(-r\ddot{\alpha}\sin\alpha - r\dot{\alpha}^2\cos\alpha\right)m + \mathbf{J}\left(r\ddot{\alpha}\cos\alpha - r\dot{\alpha}^2\sin\alpha\right)m \\ = \; & \mathbf{J}(-mg) + \mathbf{I}(-N\cos\alpha) + \mathbf{J}(-N\sin\alpha) \end{aligned} \tag{1.128}$$

This equation is equivalent to the two following scalar equations

$$\begin{aligned} \left(-r\ddot{\alpha}\sin\alpha - r\dot{\alpha}^2\cos\alpha\right)m &= -N\cos\alpha \\ \left(r\ddot{\alpha}\cos\alpha - r\dot{\alpha}^2\sin\alpha\right)m &= -N\sin\alpha - mg \end{aligned} \tag{1.129}$$

The second equation of the set 1.129 allows the normal force to be obtained

$$N = -\frac{mg}{\sin\alpha} - \frac{\cos\alpha}{\sin\alpha}mr\ddot{\alpha} + mr\dot{\alpha}^2 \tag{1.130}$$

Upon introduction of the above expression into the first equation of 1.129, one can get the following equation of motion

$$\ddot{\alpha} + \frac{g}{r}\cos\alpha = 0 \tag{1.131}$$

The initial conditions are as follows

$$\alpha|_{t=0} = -\frac{\pi}{2} \qquad \dot{\alpha}|_{t=0} = \frac{v}{r} \tag{1.132}$$

Solution of the equation of motion 1.131 with the initial conditions 1.132 if substituted into expression 1.130 yields expression for the normal force N. As long as the normal force is positive, the ball is in contact with the cylinder. $N = 0$ flags the instant of time at which the ball is loosing contact with the cylinder.

Problem 11

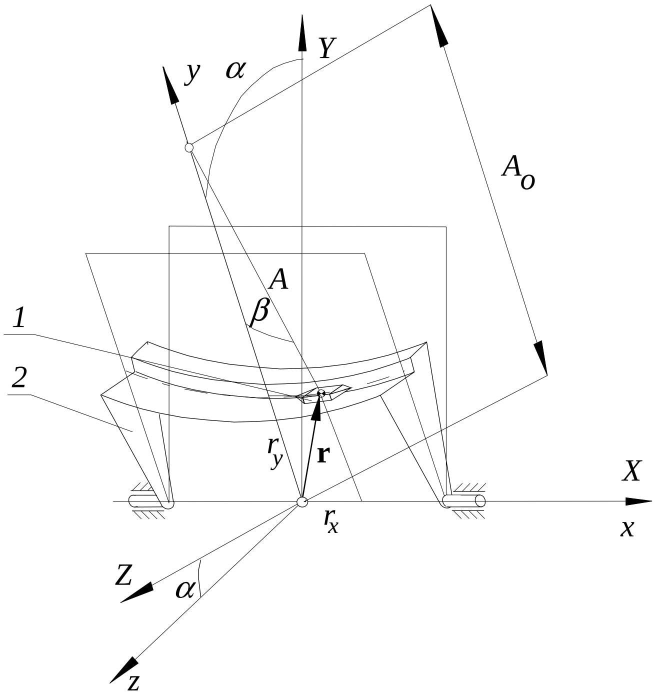


Figure 33

The relative motion of the shuttle *1* with respect to the batten *2* of a loom (Fig. 33) is given in the rotating system of coordinates x, y, z by the position vector $\mathbf{r}$.

$$\mathbf{r} = \mathbf{i}r_x + \mathbf{j}r_y \tag{1.133}$$

where:

$$r_x = A\sin\beta, \qquad r_y = A_o - A\ \cos\beta \tag{1.134}$$

The rotation of the batten *2* about axis X is determined by its angular displacement α.

Produce

1. the expressions for components of the absolute velocity and acceleration of the shuttle *1* along the absolute *(*XYZ*)* and intrinsic *(*$\mathbf{tnb}$*)* system of coordinates

2. the equation of motion of the shuttle

Solution.
According to Eq. 1.133 and Eq. 1.134 the position vector $\mathbf{r}$ is

$$\mathbf{r} = \mathbf{i}r_x + \mathbf{j}r_y = \mathbf{i}A\sin\beta + \mathbf{j}(A_o - A\cos\beta) \tag{1.135}$$

The components of the absolute velocity and acceleration may be obtained by differentiation of components of the absolute position vector $\mathbf{r}$ along inertial system of coordinates XYZ. These components may be obtained as the dot products of the vector $\mathbf{r}$ and the corresponding unit vectors $\mathbf{IJK}$.

$$\begin{aligned}
r_X &= \mathbf{I}\cdot\mathbf{r} = \mathbf{I}\cdot(\mathbf{i}r_x + \mathbf{j}r_y) = r_x = A\sin\beta \\
r_Y &= \mathbf{J}\cdot\mathbf{r} = \mathbf{J}\cdot(\mathbf{i}r_x + \mathbf{j}r_y) = \cos\alpha\ r_y = (A_o - A\cos\beta)\cos\alpha \\
r_Z &= \mathbf{K}\cdot\mathbf{r} = \mathbf{K}\cdot(\mathbf{i}r_x + \mathbf{j}r_y) = \cos(90^o - \alpha)r_y = (A_o - A\cos\beta)\sin\alpha
\end{aligned} \tag{1.136}$$

Differentiation of the above components with respect to time yields the components of the absolute velocity.

$$\begin{aligned}
\dot{r}_X &= A\dot{\beta}\cos\beta \\
\dot{r}_Y &= (A\dot{\beta}\sin\beta)\cos\alpha - (A_o - A\cos\beta)\dot{\alpha}\sin\alpha \\
\dot{r}_Z &= (A\dot{\beta}\sin\beta)\sin\alpha + (A_o - A\cos\beta)\dot{\alpha}\cos\alpha
\end{aligned} \tag{1.137}$$

Similarly, differentiation of the components of the absolute velocity allows the components of the absolute acceleration to be obtained

$$\begin{aligned}
\ddot{r}_X &= A\ddot{\beta}\cos\beta - A\dot{\beta}^2\sin\beta \\
\ddot{r}_Y &= (A\ddot{\beta}\sin\beta + A\dot{\beta}^2\cos\beta)\cos\alpha - (A\dot{\beta}sin\beta)\dot{\alpha}\sin\alpha - (A\dot{\beta}\sin\beta)\dot{\alpha}\sin\alpha \\
&\quad -(A_o - A\cos\beta)\ddot{\alpha}\sin\alpha - (A_o - A\cos\beta)\dot{\alpha}^2\cos\alpha \\
\ddot{r}_Z &= (A\ddot{\beta}\sin\beta + A\dot{\beta}^2\cos\beta)\sin\alpha + (A\dot{\beta}\sin\beta)\dot{\alpha}\cos\alpha + (A\dot{\beta}\sin\beta)\dot{\alpha}\cos\alpha \\
&\quad +(A_o - A\cos\beta)\ddot{\alpha}\cos\alpha - (A_o - A\cos\beta)\dot{\alpha}^2\sin\alpha
\end{aligned} \tag{1.138}$$

Components of the absolute acceleration along system of coordinates xyz

$$\ddot{\mathbf{r}} = \mathbf{i}a_x + \mathbf{j}a_y + \mathbf{k}a_z \tag{1.139}$$

may be obtain as a dot product of $\ddot{\mathbf{r}}$ and unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$

$$\begin{aligned}
a_x &= \ddot{\mathbf{r}}\cdot\mathbf{i} = \mathbf{I}\cdot\mathbf{i}\ddot{r}_X + \mathbf{J}\cdot\mathbf{i}\ddot{r}_Y + \mathbf{K}\cdot\mathbf{i}\ddot{r}_Z = \ddot{r}_X \\
&= A\ddot{\beta}\cos\beta - A\dot{\beta}^2\sin\beta \\
a_y &= \ddot{\mathbf{r}}\cdot\mathbf{j} = \mathbf{I}\cdot\mathbf{j}\ddot{r}_X + \mathbf{J}\cdot\mathbf{j}\ddot{r}_Y + \mathbf{K}\cdot\mathbf{j}\ddot{r}_Z = \ddot{r}_Y\cos\alpha + \ddot{r}_Z\sin\alpha \\
&= A\ddot{\beta}\sin\beta + A\dot{\beta}^2\cos\beta - (A_o - A\cos\beta)\dot{\alpha}^2 \\
a_z &= \ddot{\mathbf{r}}\cdot\mathbf{k} = \mathbf{I}\cdot\mathbf{k}\ddot{r}_X + \mathbf{J}\cdot\mathbf{k}\ddot{r}_Y + \mathbf{K}\cdot\mathbf{k}\ddot{r}_Z = -\ddot{r}_Y\sin\alpha + \ddot{r}_Z\cos\alpha \\
&= 2A\dot{\beta}\dot{\alpha}\sin\beta + (A_o - A\cos\beta)\ddot{\alpha}
\end{aligned} \tag{1.140}$$

To employ Newton's equation

$$m\ddot{\mathbf{r}} = \sum \mathbf{F} \tag{1.141}$$

let us introduce the intrinsic system of coordinates $\mathbf{t}, \mathbf{n}, \mathbf{b}$ shown in Fig. 34.

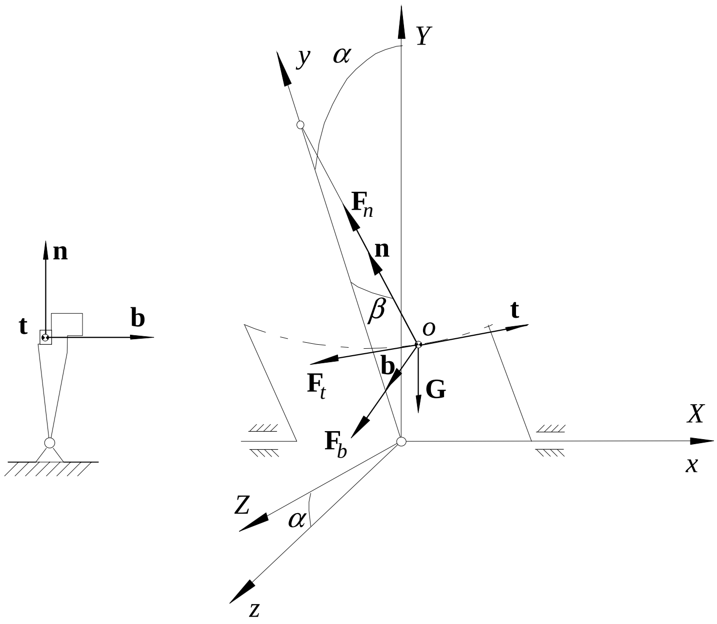


Figure 34

The unit vector $\mathbf{t}$ coincides with the relative velocity of the shuttle $\mathbf{r}'$. Hence

$$\mathbf{t} = \frac{\mathbf{r}'}{r'} \tag{1.142}$$

where

$$\mathbf{r}' = \mathbf{i}\dot{r}_x + \mathbf{j}\dot{r}_y = \mathbf{i}A\dot{\beta}\cos\beta + \mathbf{j}A\dot{\beta}\sin\beta \tag{1.143}$$

$$r' = \sqrt{A^2\dot{\beta}^2\cos^2\beta + A^2\dot{\beta}^2\sin^2\beta} = A\dot{\beta} \tag{1.144}$$

Introducing Eq's.. 1.143 and 1.144 into Eq. 1.142 one may gain

$$\mathbf{t} = \mathbf{i}\cos\,\beta + \mathbf{j}\sin\beta \tag{1.145}$$

The binormal unit vector $\mathbf{b}$ is perpendicular to the plane xy hence

$$\mathbf{b} = \mathbf{k} \tag{1.146}$$

The normal unit vector $\mathbf{n}$ may be calculated as a cross product of $\mathbf{b}$ and $\mathbf{t}$.

$$\mathbf{n} = \mathbf{b} \times \mathbf{t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ \cos\beta & \sin\beta & 0 \end{vmatrix} = -\mathbf{i}\sin\beta + \mathbf{j}\cos\beta \tag{1.147}$$

Hence, the components of acceleration along the intrinsic system are

$$\begin{aligned}
a_t &= \mathbf{t} \cdot \ddot{\mathbf{r}} = (\mathbf{i} \cos\beta + \mathbf{j} \sin\beta) \cdot (\mathbf{i}a_x + \mathbf{j}a_y + \mathbf{k}a_z) = a_x \cos\beta + a_y \sin\beta \\
&= A\ddot{\beta} - (A_o - A\cos\beta)\dot{\alpha}^2 \sin\beta \\
a_n &= \mathbf{n} \cdot \ddot{\mathbf{r}} = (-\mathbf{i} \sin\beta + \mathbf{j} \cos\beta) \cdot (\mathbf{i}a_x + \mathbf{j}a_y + \mathbf{k}a_z) = -a_x \sin\beta + a_y \cos\beta \\
&= A\dot{\beta}^2 - (A_o - A\cos\beta)\dot{\alpha}^2 \cos\beta \\
a_b &= \mathbf{b} \cdot \ddot{\mathbf{r}} = \mathbf{k} \cdot (\mathbf{i}a_x + \mathbf{j}a_y + \mathbf{k}a_z) = a_z \\
&= 2A \cdot \dot{\beta}\dot{\alpha} \sin\beta + (A_o - A\cos\beta)\ddot{\alpha}
\end{aligned} \tag{1.148}$$

Now, the Newton's equations

$$m\mathbf{a} = \mathbf{F}$$

may be written in the following form

$$\begin{aligned}
ma_b &= F_b + G_b \\
ma_t &= F_t + G_t \\
ma_n &= F_n + G_n
\end{aligned} \tag{1.149}$$

where G_b, G_t and G_n are components of the gravity force along the intrinsic system of coordinates. They are

$$\begin{aligned}
G_b &= -\mathbf{J}mg \cdot \mathbf{b} = -\,mg\mathbf{J} \cdot \mathbf{k} = -mg\cos\angle(\mathbf{J}, \mathbf{k}) = +\,mg \sin\alpha \\
G_t &= -\mathbf{J}mg \cdot \mathbf{t} = -\,mg\mathbf{J} \cdot (\mathbf{i}\cos\beta + \mathbf{j}\sin\beta) = -mg\sin\beta\cos\alpha \\
G_n &= -\mathbf{J}mg \cdot \mathbf{n} = -\,mg\mathbf{J} \cdot (-\mathbf{i}\sin\beta + \mathbf{j}\cos\beta) = -mg\cos\beta\cos\alpha
\end{aligned} \tag{1.150}$$

There is a relationship between the friction force F_t and the interaction forces F_b and F_n. Namely

$$F_t = \mu_b F_b + \mu_n F_n \tag{1.151}$$

where μ_b and μ_n are the friction coefficients between the walls and the shuttle.

Introduction of 1.151 into 1.149 yields

$$\begin{aligned}
ma_b &= F_b + G_b \\
ma_t &= \mu_b F_b + \mu_n F_n + G_t \\
ma_n &= F_n + G_n
\end{aligned} \tag{1.152}$$

The above equations are linear with respect to the interaction forces F_b and F_n.Therefore they can be easily eliminated from the equations 1.152.

$$ma_t = \mu_b (ma_b - G_b) + \mu_n (ma_n - G_n) + G_t \tag{1.153}$$

Introduction of 1.148 and 1.150 into 1.153 yields the differential equation that govern motion of the shuttle.

$$\begin{aligned}
& m\left(A\ddot{\beta} - (A_o - A\cos\beta)\dot{\alpha}^2 \sin\beta\right) \\
= \;& \mu_b(m\left(2A \cdot \dot{\beta}\dot{\alpha}\sin\beta + (A_o - A\cos\beta)\ddot{\alpha}\right) - mg\sin\alpha) \\
& +\mu_n(m\left(A\dot{\beta}^2 - (A_o - A\cos\beta)\dot{\alpha}^2\cos\beta\right) + mg\cos\beta\cos\alpha) - mg\sin\beta\cos\alpha
\end{aligned}$$

Its solution $\beta(t)$ approximate motion of the shuttle.

Problem 12

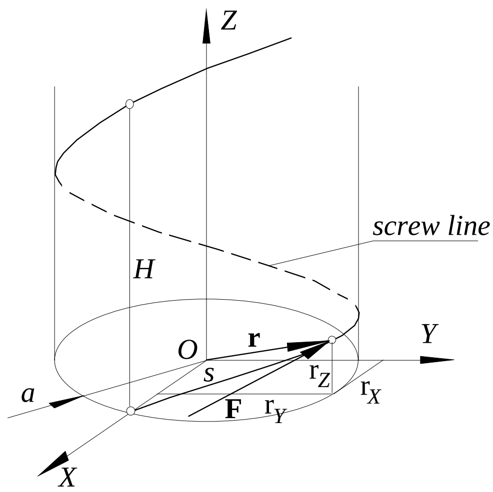


Figure 35

The bead of mass m, which can be considered as a particle, moves along the screw line shown in Fig. 35. Its motion with respect to the inertial space XYZ is given by the following position vector

$$\mathbf{r} = \mathbf{I}a\cos\omega t + \mathbf{J}a\sin\omega t + \mathbf{K}bt \tag{1.154}$$

Produce expression for
1. the interaction force between the spiral slide and the bead.
Answer:
$\mathbf{R} = \mathbf{I}(-ma\omega^2\cos\omega t + \frac{mgab\omega}{a^2\omega^2+b^2}\sin\omega t) + \mathbf{J}(-ma\omega^2\sin\omega t + \frac{-mgab\omega}{a^2\omega^2+b^2}\cos\omega t) + \mathbf{K}(\frac{mga^2\omega^2}{a^2\omega^2+b^2})$
2. the force F which must be applied to the bead to ensure its motion according to the equation 1.154
Answer:
$\mathbf{F} = \mathbf{I}(-\frac{mgab\omega}{a^2\omega^2+b^2}\sin\omega t) + \mathbf{J}(\frac{mgab\omega}{a^2\omega^2+b^2}\cos\omega t) + \mathbf{K}(\frac{mgb^2}{a^2\omega^2+b^2})$
3. the work produced by individual forces acting on the bead for $0 < t < \frac{2\pi}{\omega}$
Answer:
$W_R = 0 \qquad W_F = \frac{2\pi mgb}{\omega} \qquad W_G = -\frac{2\pi mgb}{\omega}$
4. the kinetic energy of the particle as a function of time
Answer:
$T = \frac{1}{2}m(a^2\omega^2 + b^2)$

Problem 13

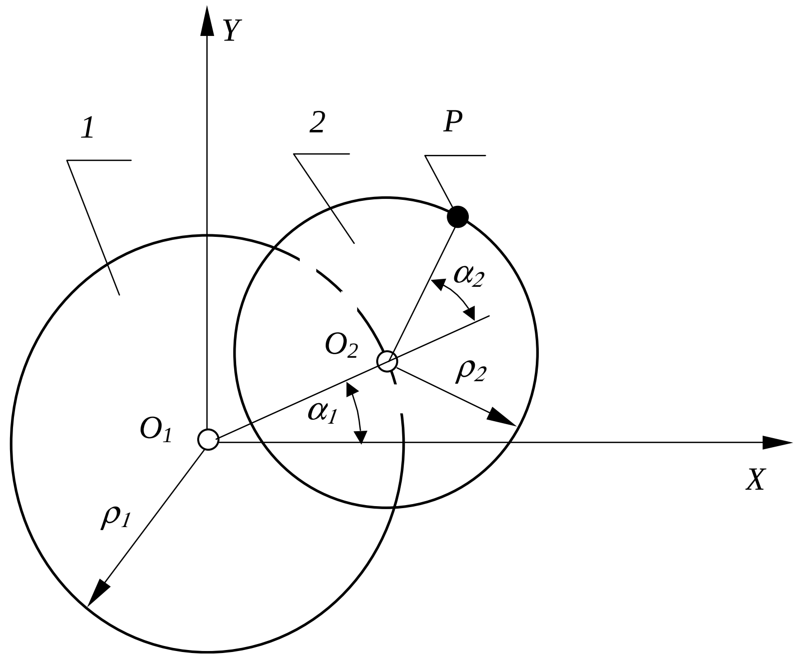


Figure 36

Figure 36 shows the sketch of a Ferris wheel. Its base 1, of radius ρ_1, rotates about the horizontal axis Z of the inertial system of coordinates XYZ. Its instantaneous position is determined by the angular displacement α_1. The wheel 2, of radius ρ_2, rotates about axis parallel to Z through the point O_2. Its relative rotation is determined by the angular position α_2. A person that can be treated as a particle of the mass m is seating at the wheel 2. Motion of the Ferris wheel is restricted to the vertical plane XY.

Produce

1. the expression for the work produced by the gravity force acting on the particle P

Answer:

$W = -mg(\rho_1 \sin\alpha_1 + \rho_2 \sin(\alpha_1 + \alpha_2))$

2. the expression for the kinetic energy of the particle P

Answer:

$T = \frac{1}{2}m\left(\rho_1^2\dot{\alpha}_1^2 + \rho_2^2\dot{\alpha}_1^2 + \rho_2^2\dot{\alpha}_2^2 + 2\rho_2^2\dot{\alpha}_1\dot{\alpha}_2 + 2\rho_1\rho_2\dot{\alpha}_1\dot{\alpha}_2\cos\alpha_2 + 2\rho_1\rho_2\dot{\alpha}_1^2\cos\alpha_2\right)$

3. the expression for the components of the interaction force between the particle P and the wheel

Answer:

$R_X = ma_{PX}$

$R_Y = ma_{PY} + mg$

where

$a_{PX} = -\rho_1\ddot{\alpha}_1\sin\alpha_1 - \rho_1\dot{\alpha}_1^2\cos\alpha_1 - \rho_2(\ddot{\alpha}_1+\ddot{\alpha}_2)\sin(\alpha_1+\alpha_2) - \rho_2(\dot{\alpha}_1+\dot{\alpha}_2)^2\cos(\alpha_1+\alpha_2)$

$a_{PY} = +\rho_1\ddot{\alpha}_1\cos\alpha_1 - \rho_1\dot{\alpha}_1^2\sin\alpha_1 + \rho_2(\ddot{\alpha}_1+\ddot{\alpha}_2)\cos(\alpha_1+\alpha_2) - \rho_2(\dot{\alpha}_1+\dot{\alpha}_2)^2\sin(\alpha_1+\alpha_2)$

Problem 14

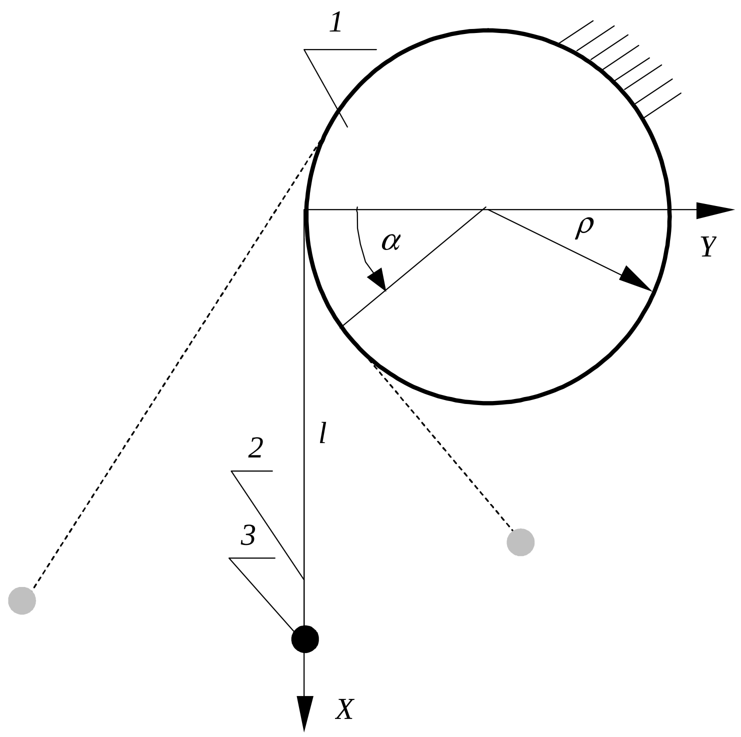


Figure 37

The particle 3 is suspended from the cylinders 1 of radius ρ by means of the massless rope 2 of length l (see Fig. 37). Its instantaneous position can be determined by the angular displacement α. Motion of the particle is restricted to the vertical plane XY of an inertial system of coordinates.

Produce

1. the differential equation of motion of the particle 3

Answer:

$a_X - a_Y \cot\alpha - g = 0$

where

$a_X = -l\ddot{\alpha}\sin\alpha - l\dot{\alpha}^2\cos\alpha + \varrho\dot{\alpha}^2\sin\alpha + \varrho\alpha\ddot{\alpha}\sin\alpha + \varrho\alpha\dot{\alpha}^2\cos\alpha$

$a_Y = +l\ddot{\alpha}\cos\alpha - l\dot{\alpha}^2\sin\alpha - \varrho\dot{\alpha}^2\cos\alpha - \varrho\alpha\ddot{\alpha}\cos\alpha + \varrho\alpha\dot{\alpha}^2\sin\alpha$

2. the expression for the tension in the rope as a function of the angular position α

Answer:

$T = -ma_Y\frac{1}{\sin\alpha}$

Problem 15

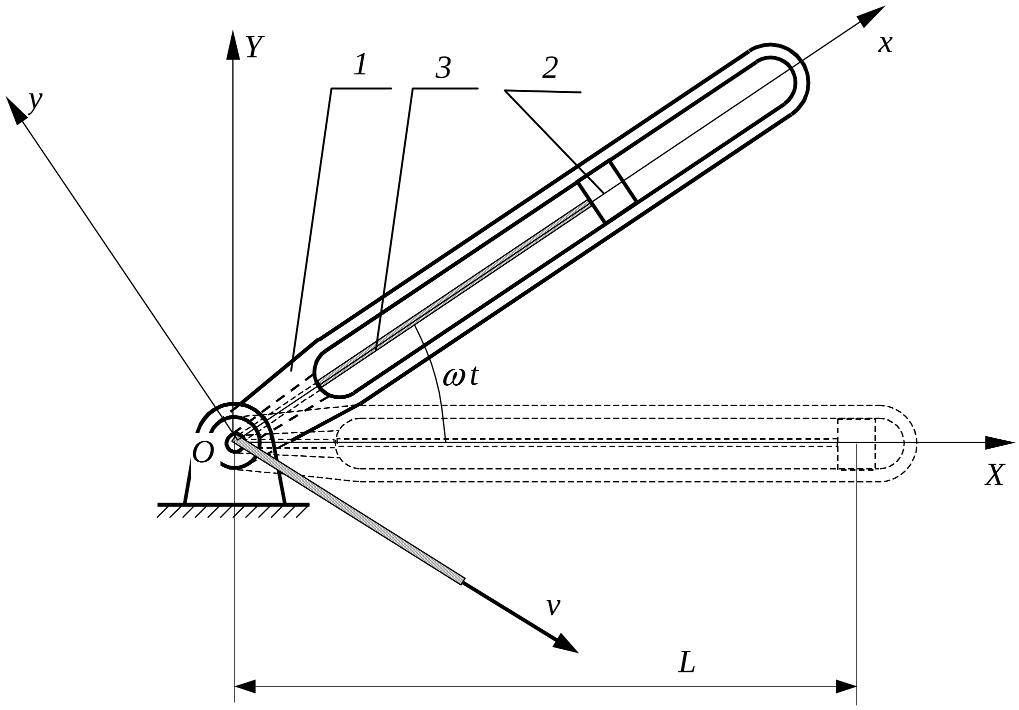


Figure 38

The slider 2, shown in Fig. 38, can be moved inward by means of the string 3 while the slotted arm 1 rotates about the horizontal axis Z of the inertial system of coordinates XYZ. The relative velocity v of the slider with respect to the arm is constant and the absolute angular velocity of the arm ω is constant too. The slider 2 can be considered as a particle of mass m. When the arm coincides with the axis X the slider is L distant from the origin O. The system of coordinates xyz is rigidly attached to the arm 1.

Produce:

1. the expression for the components of the absolute linear velocity of the slider 2 along the system of coordinates xyz

Answer:

$$\mathbf{v}_s = \mathbf{i}(-v) + \mathbf{j}\omega(L - vt)$$

2. the expression for the components of the absolute linear acceleration of the slider 2 along the system of coordinates xyz

Answer:

$$\mathbf{a}_s = \mathbf{i}(-\omega^2(L - vt)) + \mathbf{j}(-2\omega v)$$

3. the expression for the components of the absolute linear acceleration of the slider 2 along the system of coordinates XYZ

Answer:

$$\mathbf{a}_s = \mathbf{I}(-\omega^2(L - vt)\cos\omega t + 2\omega v \sin\omega t) + \mathbf{J}(-\omega^2(L - vt)\sin\omega t - 2\omega v \cos\omega t)$$

4. the expression for the interaction force between the slider 2 and the arm 1

Answer:
$N = mg\cos\omega t - 2m\omega v$
5. the expression for the tension in the string 3
Answer:
$T = m\omega^2(L - vt) - mg\sin\omega t$

Problem 16

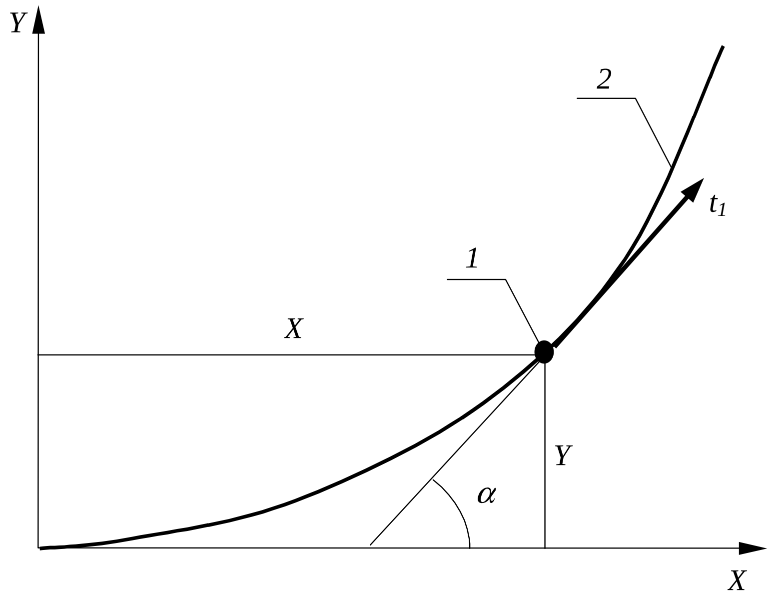


Figure 39

A skier that can be treated as the particle 1 of mass m (see Fig. 39) is skiing down the slope 2. The shape of slope is determined by the following equation

$$Y = aX^2$$

There is a dry friction between the skier 1 and the slope 2. The friction coefficient is μ.

Produce:
1. the expression for the components of the tangential unit vector $\mathbf{t}_1$
Answer:
$\mathbf{t}_1 = \mathbf{I}\frac{1}{\sqrt{1+4a^2X^2}} + \mathbf{J}\frac{2aX}{\sqrt{1+4a^2X^2}}$
2. the expression for the gradient α.
Answer:
$\alpha = \arccos\frac{1}{\sqrt{1+4a^2X^2}}$
3. the free body diagram for the skier 1.
4. the differential equations of motion of the skier
Answer:
$2ma\dot{X}^2 + 2amX\ddot{X} = -mg + \frac{m\ddot{X}(\cos\alpha+\mu\sin\alpha)}{\mu\cos\alpha-\sin\alpha}$
where $\cos\alpha = \frac{1}{\sqrt{1+4a^2X^2}}$ $\quad \sin\alpha = \frac{2aX}{\sqrt{1+4a^2X^2}}$

Problem 17

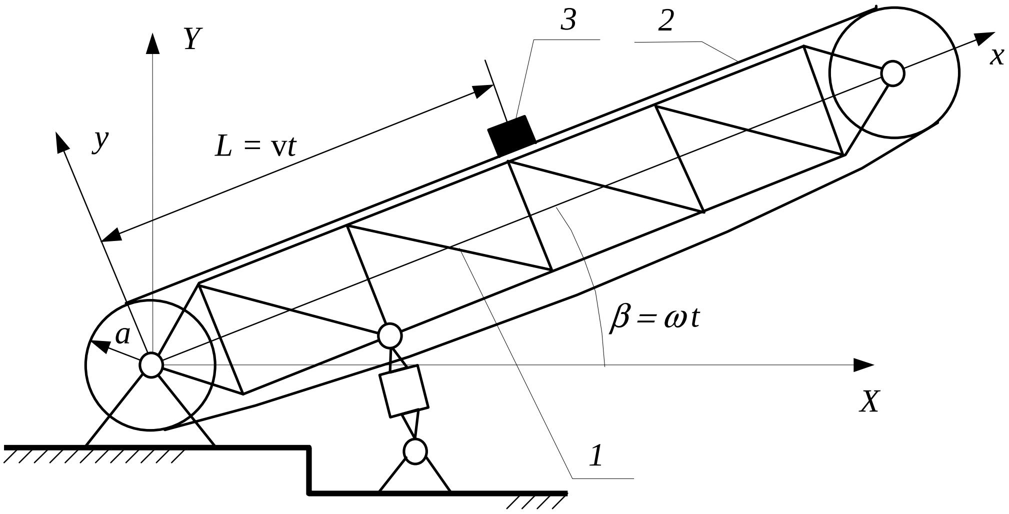


Figure 40

The belt *2* of the belt conveyor *1* shown in Fig. 40 moves with a constant velocity v. Simultaneously, its boom is being raised with a constant angular velocity ω. The object *3* that can be considered as a particle of mass m is motionless with respect to the belt *2*.

Produce:

1. The expression for the absolute linear velocity of the object *3*
2. The expression for the tangential, normal and binormal unit vectors associated with the absolute trajectory of motion of the object *3*
3. The expression for the absolute linear acceleration of the object *3*
4. The expression for the interaction forces between the object *3* and the belt *2*

Solution
1.

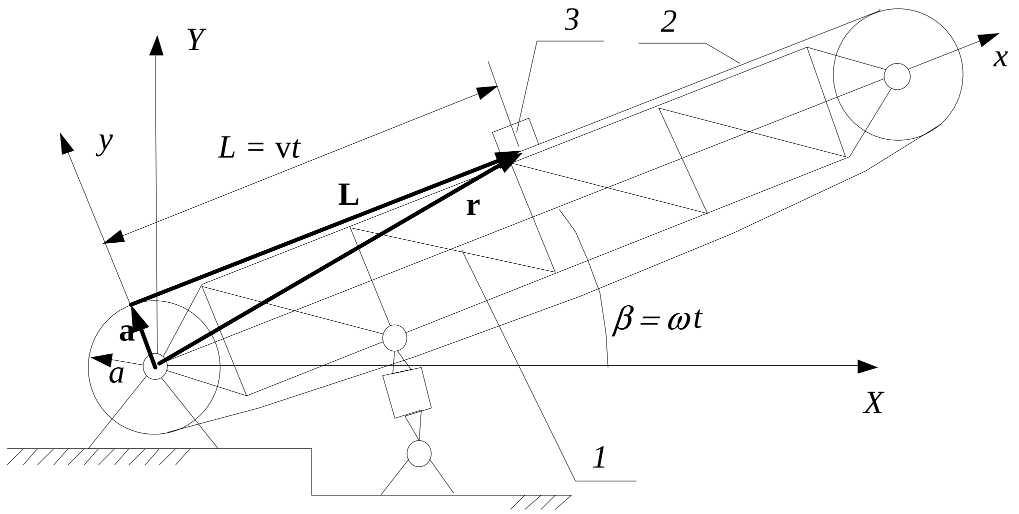


Figure 41

The absolute position vector of the object 3 (see Fig. 41) is

$$\mathbf{r} = \mathbf{a} + \mathbf{L} = \mathbf{j}(a) + \mathbf{i}(vt) \tag{1.155}$$

It components along the inertial system of coordinates XYZ are

$$\begin{aligned} r_X &= \mathbf{I} \cdot \mathbf{r} = \mathbf{I} \cdot (\mathbf{j}(a) + \mathbf{i}(vt)) = -a \sin \omega t + vt \cos \omega t \\ r_Y &= \mathbf{J} \cdot \mathbf{r} = \mathbf{J} \cdot (\mathbf{j}(a) + \mathbf{i}(vt)) = a \cos \omega t + vt \sin \omega t \end{aligned} \tag{1.156}$$

The components of the absolute velocity $\mathbf{v}$ of the object are

$$\begin{aligned} v_X &= \dot{r}_X = -a\omega \cos \omega t + v \cos \omega t - vt\omega \sin \omega t \\ v_Y &= \dot{r}_Y = -a\omega \sin \omega t + v \sin \omega t + vt\omega \cos \omega t \end{aligned} \tag{1.157}$$

Magnitude of the absolute velocity is

$$v = \sqrt{v_X^2 + v_Y^2} = \sqrt{(v - a\omega)^2 + (v\omega t)^2} \tag{1.158}$$

2.

The tangential unit vector associated with the absolute trajectory is

$$\begin{aligned} \mathbf{t}_1 = \frac{\mathbf{v}}{v} = \frac{1}{\sqrt{(v-a\omega)^2 + (v\omega t)^2}} &\mathbf{I}(-a\omega \cos \omega t + v \cos \omega t - vt\omega \sin \omega t) \\ &+ \mathbf{J}(-a\omega \sin \omega t + v \sin \omega t + vt\omega \cos \omega t) = \\ = \mathbf{I}\,(t_{1X}) + \mathbf{J}\,(t_{1Y}) \end{aligned} \tag{1.159}$$

Since the absolute trajectory is placed in the the plane XY, the binormal unit vector is perpendicular to this plane

$$\mathbf{b}_1 = \mathbf{K}\,(1) \tag{1.160}$$

Hence, the normal unit vector is

$$\begin{aligned}
\mathbf{t}_1 \times \mathbf{n}_1 &= \mathbf{b}_1 \\
(\mathbf{t}_1 \times \mathbf{n}_1) \times \mathbf{t}_1 &= \mathbf{b}_1 \times \mathbf{t}_1 \\
-\mathbf{t}_1 \times (\mathbf{t}_1 \times \mathbf{n}_1) &= \mathbf{b}_1 \times \mathbf{t}_1 \\
-\mathbf{t}_1 \cdot (\mathbf{n}_1 \cdot \mathbf{t}_1) + \mathbf{n}_1 \cdot (\mathbf{t}_1 \cdot \mathbf{t}_1) &= \mathbf{b}_1 \times \mathbf{t}_1 \\
\mathbf{n}_1 &= \mathbf{b}_1 \times \mathbf{t}_1 \\
\mathbf{n}_1 &= \mathbf{b}_1 \times \mathbf{t}_1 = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 0 & 0 & 1 \\ t_{1X} & t_{1Y} & 0 \end{vmatrix} = \\
&= \mathbf{I}\,(-t_{1Y}) + \mathbf{J}\,(t_{1X})
\end{aligned} \tag{1.161}$$

3.

The absolute acceleration as the fist derivative of the absolute velocity vector is

$$\mathbf{a} = \mathbf{I}a_X + \mathbf{J}a \tag{1.162}$$

where

$$\begin{aligned}
a_X &= \dot{v}_X = -2v\omega \sin\omega t - vt\omega^2 \cos\omega t + a\omega^2 \sin\omega t \\
a_Y &= \dot{v}_Y = 2v\omega \cos\omega t - vt\omega^2 \sin\omega t - a\omega^2 \cos\omega t
\end{aligned} \tag{1.163}$$

4.

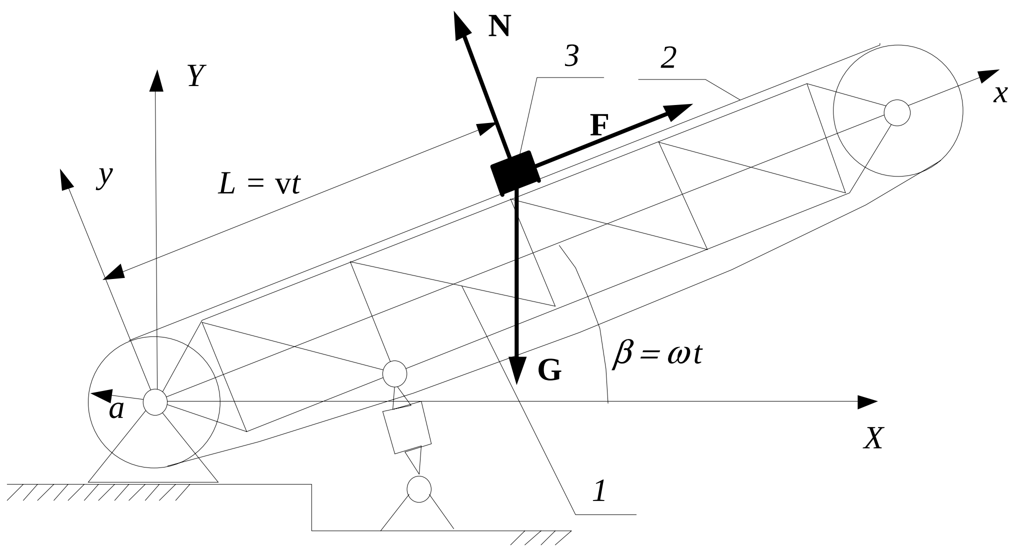


Figure 42

Application of the Newton's law to the FBD shown in Fig. 42 yields

$$\begin{aligned}
ma_X &= -N \sin\omega t + F \cos\omega t \\
ma_Y &= N \cos\omega t + F \sin\omega t - G
\end{aligned} \tag{1.164}$$

Hence, the interaction forces are

$$\begin{aligned} N &= -ma_X \sin\omega t + ma_Y \cos\omega t + G\cos\omega t \\ F &= ma_X \cos\omega t + ma_Y \sin\omega t + G\sin\omega t \end{aligned} \tag{1.165}$$

Chapter 2

PLANE DYNAMICS OF A RIGID BODY

2.1 PLANE KINEMATICS OF A RIGID BODY

DEFINITION: A body which by assumption is not deformable and therefore the distances between its points remains unchanged, regardless of forces acting on the body, is called *rigid body.*

DEFINITION: If there exists such a plane XY of an inertial space XYZ that trajectories of all points of the body considered are parallel to this plane, motion of the body is called *plane motion* **and plane XY is called** *plane of motion.*

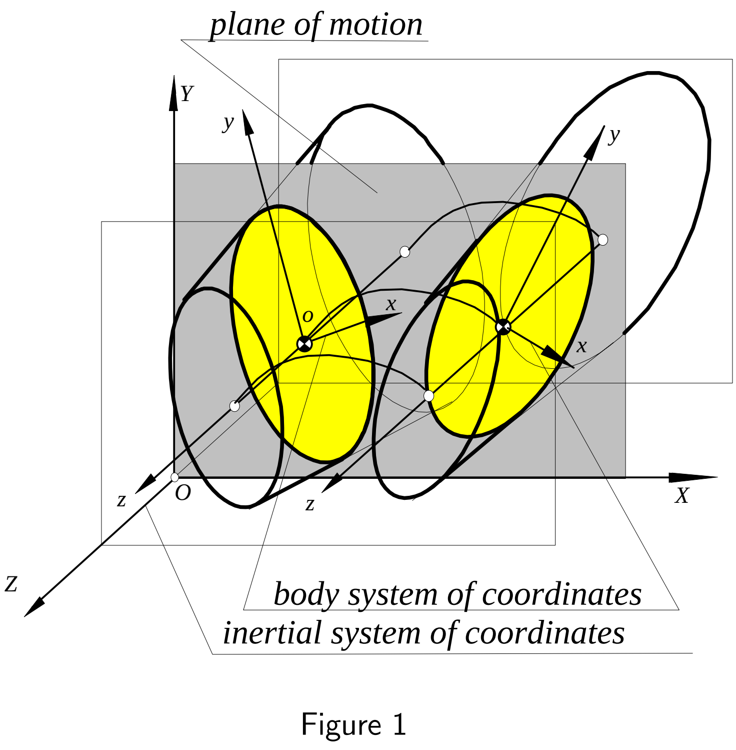


Figure 1

In this chapter we shall assume that such a plane of motion exists.

To analyze motion of a rigid body we attach to the body a system of coordinates xyz at an arbitrarily chosen point o (see Fig.1). Such a system of coordinates is called *body system of coordinates.* Axis z of the body system of coordinates is chosen to be parallel to the axis Z of the inertial system of coordinates as is shown in Fig 1. Hence, the plane xy of the body system of coordinates is always parallel to the plane XY. Furthermore, without loosing the generality of consideration, we can assume that the origin of the body system of coordinates and origin of the inertial system of coordinates are in the same plane. Hence, motion of the rigid body can be uniquely determined by motion of its body system of coordinates. The body system of coordinates, in a general case, can not be classified as the inertial one

*2.1.1 **Non-inertial (moving) systems of coordinates***

DEFINITION: System of coordinates which can not be classified as inertial is called *non-inertial system of coordinates.*

The non-inertial systems of coordinates are denoted by lower characters (e.g. xy) to distinguish them from inertial ones which are usually denoted by upper characters (e.g. XY).

DEFINITION: If a non-inertial system of coordinates does not rotate (its axes xy are always parallel to an inertial system) **the system is called** *translating system of coordinates* (Fig. 2 a).

DEFINITION: If the origin of a non-inertial system of coordinates coincides always the origin of an inertial system of coordinates, the non-inertial system is called *rotating system of coordinates* (Fig. 2b)

In a general case a non-inertial system of coordinates may translate and rotate.

DEFINITION: System of coordinates which can translate and rotate is called *translating and rotating system of coordinates* (Fig. 2c)

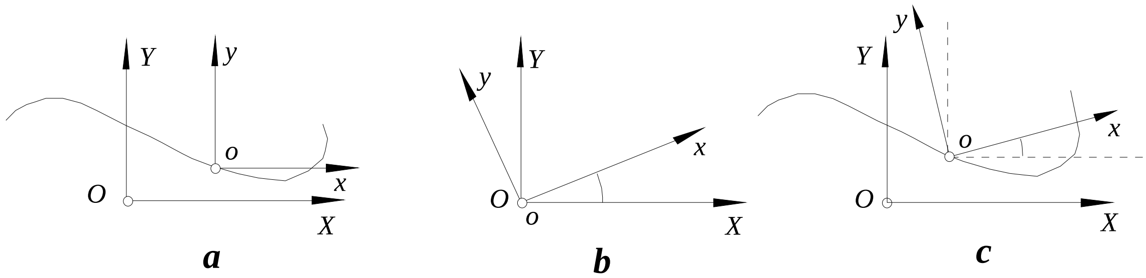


Figure 2

Translating system of coordinates

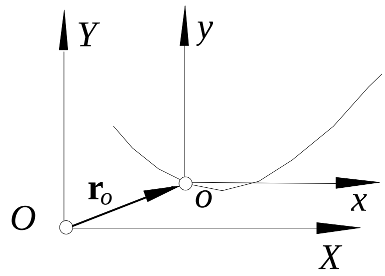


Figure 3

Since axes of the translating system of coordinates are always parallel to the axes of the inertial system of coordinates, the components of a vector along the non-inertial system of coordinates are equal to the components of the inertial system of coordinates. To determine motion of the translating system of coordinates it is sufficient to introduce one position vector $\mathbf{r}_o$ (Fig. 3).

Rotating system of coordinates

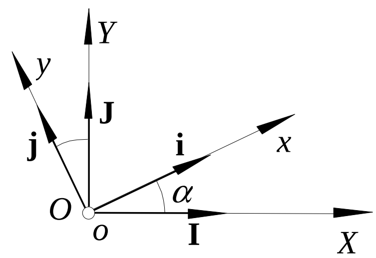


Figure 4

Since axes of the rotating system of coordinates change their angular position with respect to the inertial system of coordinates, the components of a vector along the inertial and the rotating system of coordinates are not equal to each other. Therefore it is essential to develop a procedure that allows the components of a vector along the inertial system of coordinates to be produced if the vector is given by its components along the rotating system of coordinates.

Matrix of direction cosines

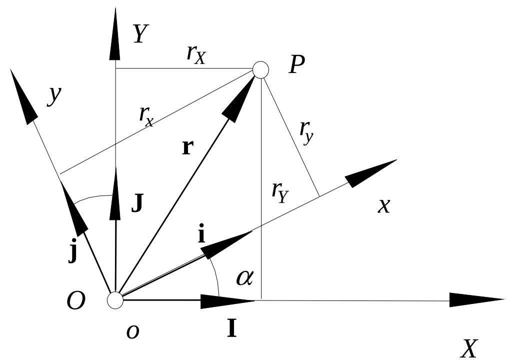


Figure 5

Let us assume that the vector **r** (Fig. 5) is given by its components along the rotating system of coordinates *xyz.*

$$\mathbf{r} = \mathbf{i}r_x + \mathbf{j}r_y \tag{2.1}$$

Components of the vector **r** along the inertial system of coordinates XY may be obtained as the scalar products of the vector **r** and the unit vectors **IJ**.

$$\begin{aligned} r_X &= \mathbf{r}\cdot\mathbf{I} = r_x\mathbf{i}\cdot\mathbf{I} + r_y\mathbf{j}\cdot\mathbf{I} = r_x\cos\angle\mathbf{iI} + r_y\cos\angle\mathbf{jI} \\ r_Y &= \mathbf{r}\cdot\mathbf{J} = r_x\mathbf{i}\cdot\mathbf{J} + r_y\mathbf{j}\cdot\mathbf{J} = r_x\cos\angle\mathbf{iJ} + r_y\cos\angle\mathbf{jJ} \end{aligned} \tag{2.2}$$

The last relationship can be written in the following matrix form

$$\begin{bmatrix} r_X \\ r_Y \end{bmatrix} = \begin{bmatrix} \cos\angle\mathbf{iI} & \cos\angle\mathbf{jI} \\ \cos\angle\mathbf{iJ} & \cos\angle\mathbf{jJ} \end{bmatrix} \begin{bmatrix} r_x \\ r_y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} r_x \\ r_y \end{bmatrix} = [C_{r\to i}] \begin{bmatrix} r_x \\ r_y \end{bmatrix} \tag{2.3}$$

The transfer matrix $[C_{r\to i}]$ is called *matrix of direction cosines.*

If we assume that the vector **r** is given by its components along the inertial system of coordinates and repeat the above derivation, we are getting

$$\begin{bmatrix} r_x \\ r_y \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} r_X \\ r_Y \end{bmatrix} = [C_{i\to r}] \begin{bmatrix} r_X \\ r_Y \end{bmatrix} \tag{2.4}$$

Hence one can conclude that the transfer matrix of the direction cosines is equal to the inverse matrix of the direction cosines

$$[C_{r\to i}]^{-1} = [C_{r\to i}]^T = [C_{i\to r}] \tag{2.5}$$

Another useful relationship should be noticed from Fig. 6. Cosine of angle between two unit vectors e.g. **i** and **J** is equal to the component of one on the other.

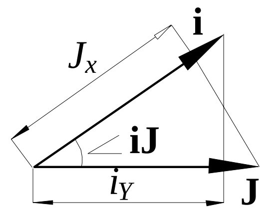


Figure 6

$$\cos\angle\mathbf{iJ} = \frac{i_Y}{i} = i_Y = \frac{J_x}{J} = J_x \tag{2.6}$$

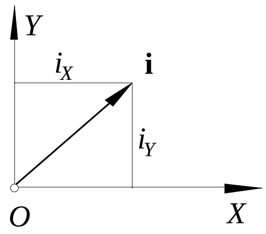


Figure 7

From Fig. 7 one can see that components i_X, i_Y ,which are equal to corresponding direction cosines, fulfil the following relationship.

$$i_X^2 + i_Y^2 = \cos^2\angle\mathbf{iI} + \cos^2\angle\mathbf{iJ} = 1 \tag{2.7}$$

Angular velocity and angular acceleration

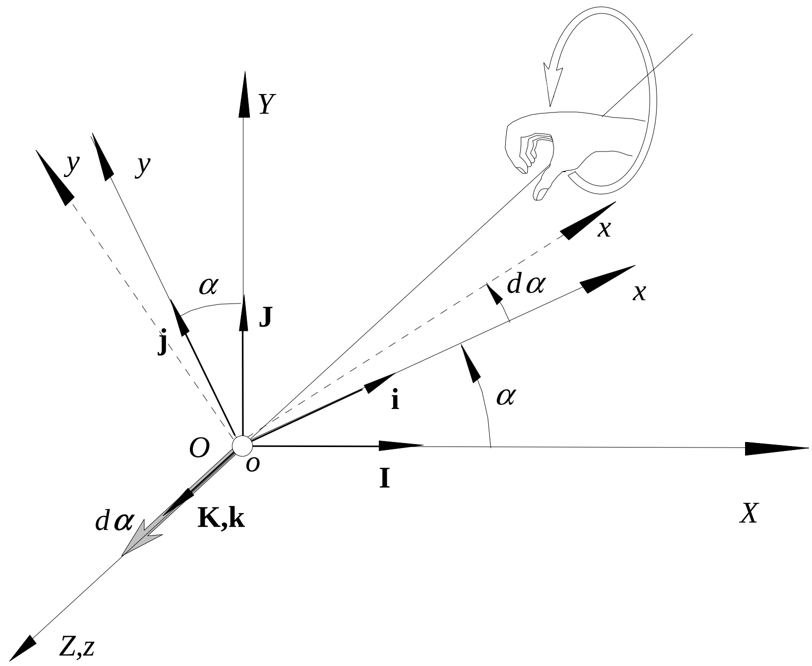


Figure 8

It is easy to show that the infinitesimal angular displacement $d\alpha$ can be considered as a vectorial magnitude. The vector of the infinitesimal angular displacement is perpendicular to the plane of rotation (XY) and its sense is determined by the *right-hand rule* (see Fig. 8). Such vectors which have determined only direction and sense only, are called *free vectors* to distinguish them from *linear vectors* (sense and line of action is determined) and *position vectors* (position of its tail, direction and sense is determined). The angular displacement is considered positive if the sense of its vectorial representation coincides with the positive direction of the axis z.

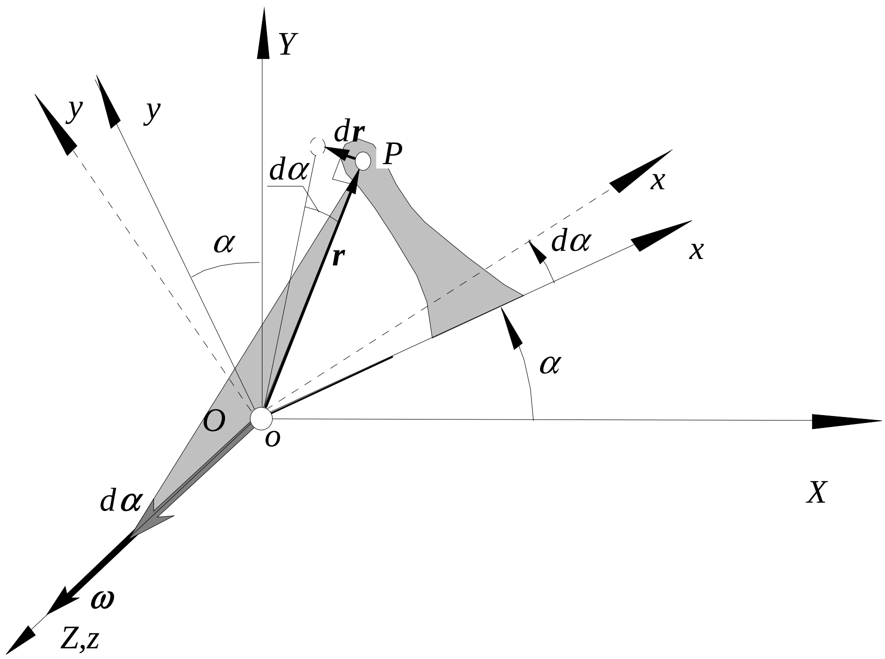


Figure 9

Now, let us consider point P that is fixed with respect the xyz system of coordinates. Its position is determined by the position vector $\mathbf{r}$ (Fig. 9). Let $d\boldsymbol{\alpha}$ be an infinitesimal angular displacement of the system of coordinates xyz. Hence, the point P at the instance considered moves along the circle of radius r. The infinitesimal increment $d\mathbf{r}$ of vector $\mathbf{r}$ is tangential to the circle. Its scalar magnitude is

$$dr = r\ d\alpha \tag{2.8}$$

Since

1. vector $d\mathbf{r}$ is perpendicular to the plane formed by vectors $d\boldsymbol{\alpha}$ and $\mathbf{r}$
2. magnitude of the vector $d\mathbf{r}$ is $r\ d\alpha \sin 90^o$
3. vectors $d\boldsymbol{\alpha}$, $\mathbf{r}$, $d\mathbf{r}$ forms the right handed system of vectors

the vector $d\mathbf{r}$ can be considered as the vector product of vector $d\boldsymbol{\alpha}$ and $\mathbf{r}$.

$$d\mathbf{r} = d\boldsymbol{\alpha} \times \mathbf{r} \tag{2.9}$$

The velocity of the point P is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\boldsymbol{\alpha}}{dt} \times \mathbf{r} \tag{2.10}$$

DEFINITION: The vector $\frac{d\boldsymbol{\alpha}}{dt}$ is called *vector of the absolute angular velocity* $\boldsymbol{\omega}$ **of the rotating system of coordinates** xyz.

$$\boldsymbol{\omega} = \frac{d\boldsymbol{\alpha}}{dt} \tag{2.11}$$

According to 2.11, the vector of the angular velocity possesses all properties of the vector $d\boldsymbol{\alpha}$.
The velocity of P can be expressed as follow.

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{2.12}$$

The angular acceleration is defined as the first derivative of the vector of the angular velocity with respect to time.

$$\boldsymbol{\varepsilon} = \frac{d\boldsymbol{\omega}}{dt} = \dot{\boldsymbol{\omega}} \tag{2.13}$$

Derivative of a vector expressed by means of components along a rotating system of coordinates

Motion of the rotating system of coordinates xy with respect to the inertial one can be defined by the angular displacement α or, alternatively, the rotational motion can be determined by its initial position and the vector of its angular velocity $\boldsymbol{\omega}$.

Let $\boldsymbol{\omega}$ be the absolute angular velocity of the rotating system of coordinates xyz (see Fig. 10).

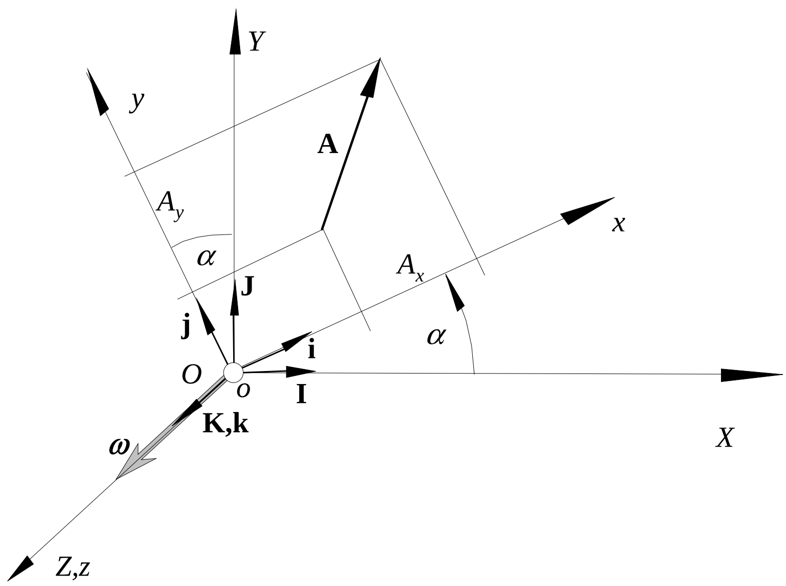


Figure 10

Consider a vector $\mathbf{A}$ that is given by its components along the rotating system of coordinates xy.

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y \tag{2.14}$$

Let us differentiate this vector with respect to time.

$$\dot{\mathbf{A}} = \frac{d}{dt}(\mathbf{i}A_x + \mathbf{j}A_y) \tag{2.15}$$

Hence

$$\dot{\mathbf{A}} = \mathbf{i}\dot{A}_x + \mathbf{j}\dot{A}_y + \dot{\mathbf{i}}A_x + \dot{\mathbf{j}}A_y \tag{2.16}$$

The first two terms represent vector which can be obtained by direct differentiating of the components A_x, A_y with respect to time. This vector will be denoted by $\mathbf{A}'$

$$\mathbf{A}' = \mathbf{i}\dot{A}_x + \mathbf{j}\dot{A}_y \tag{2.17}$$

Thus

$$\dot{\mathbf{A}} = \mathbf{A}' + \dot{\mathbf{i}}A_x + \dot{\mathbf{j}}A_y \tag{2.18}$$

According to the definition of the vector derivative, the first derivative of the unit vector $\mathbf{i}$ is the ratio of the infinitesimal vector increment $d\mathbf{i}$ and dt (Fig. 11).

$$\dot{\mathbf{i}} = \frac{d\mathbf{i}}{dt} = \frac{\mathbf{v}_i dt}{dt} = \mathbf{v}_i \tag{2.19}$$

were $\mathbf{v}_i$ is a velocity of the head of the vector $\mathbf{i}$ (see Fig.11).

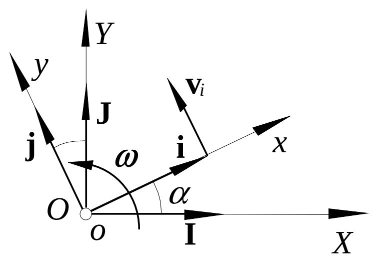


Figure 11

But, according to Eq. 2.12

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{i} \tag{2.20}$$

Hence

$$\dot{\mathbf{i}} = \boldsymbol{\omega} \times \mathbf{i} \tag{2.21}$$

Similarly

$$\dot{\mathbf{j}} = \boldsymbol{\omega} \times \mathbf{j} \tag{2.22}$$

Introducing the above expressions into Eq. 2.18 we have

$$\dot{\mathbf{A}} = \mathbf{A}' + \boldsymbol{\omega} \times \mathbf{i}A_x + \boldsymbol{\omega} \times \mathbf{j}A_y = \mathbf{A}' + \boldsymbol{\omega} \times (\mathbf{i}A_x + \mathbf{j}A_y) \tag{2.23}$$

and eventually one may gain

$$\dot{\mathbf{A}} = \mathbf{A}' + \boldsymbol{\omega} \times \mathbf{A} \tag{2.24}$$

where:

$\mathbf{A}' = \mathbf{i}\dot{A}_x + \mathbf{j}\dot{A}_y$

$\boldsymbol{\omega}$ - is the absolute angular velocity of the rotating system of coordinates xy along which the vector $\mathbf{A}$ was resolved to produce vector $\mathbf{A}'$

$\mathbf{A}$ - is the differentiated vector.

The last formula provides the rule for differentiation of a vector that is resolved along a non-inertial system of coordinates. It can be applied to any vector (position vectors, linear vectors, free vectors).

Translating and rotating system of coordinates

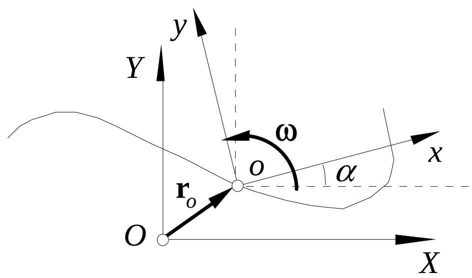


Figure 12

Motion of the translating and rotating system of coordinates xyz shown in Fig. 12, can be uniquely determined by means of the two vectors $\mathbf{r}_o$ and $\boldsymbol{\omega}$. The position vector $\mathbf{r}_o$ determines motion of its origin with respect to the inertial system of coordinates XYZ and the vector of the angular velocity $\boldsymbol{\omega}$ determines its rotation.

2.1.2 Motion of rigid body

Rotation of rigid body about fixed axis

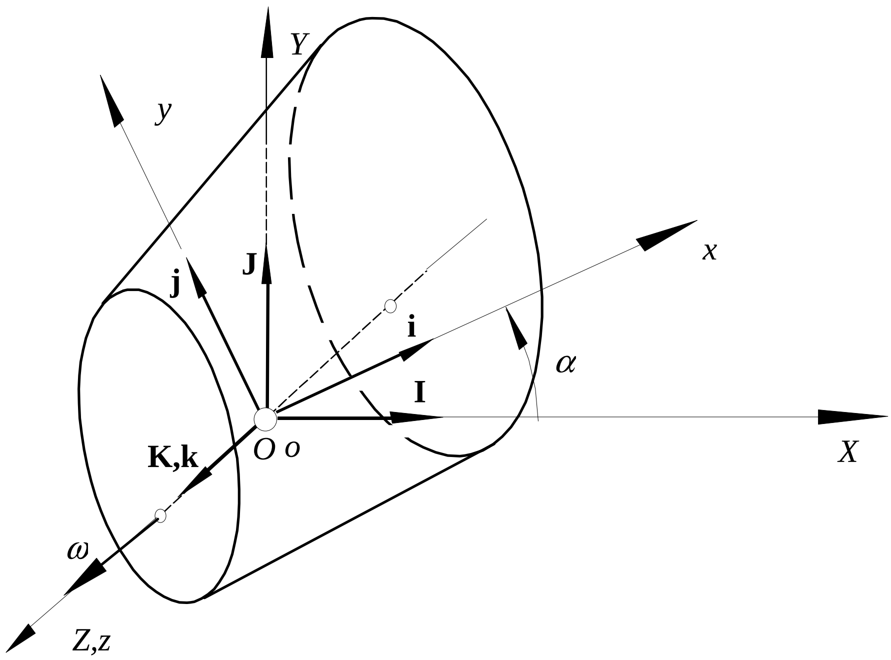


Figure 13

DEFINITION: If there exists such axis of a rigid body that does not move with respect to the inertial system of coordinates, it is said that the body performs *rotation* about this axis and the axis is called *axis of rotation.*

Let us assume that the system of coordinates xyz shown in Fig. 13 rotates about axis Z of the inertial system of coordinates XYZ. Its rotational motion can be uniquely determined by the angular displacement α or, alternatively, by the angular velocity ω. Let us attach to this system of coordinates a rigid body. According to the above definition the body performs rotation about axis Z.

DEFINITION: The angular velocity of the system of coordinate xyz **is called** *angular velocity of the rigid body* **that is attached to the system of coordinates.**

The vector ω of the angular velocity of the rigid body is

$$\boldsymbol{\omega} = \mathbf{K}\dot{\alpha} = \mathbf{k}\dot{\alpha} \tag{2.25}$$

The angular acceleration of the body is the time derivative of the vector of the angular velocity. Since the vector $\boldsymbol{\omega}$ is always perpendicular to the plane of motion of body, its derivative is.

$$\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}} = \mathbf{K}\frac{d}{dt}\dot{\alpha} = \mathbf{K}\ddot{\alpha} \tag{2.26}$$

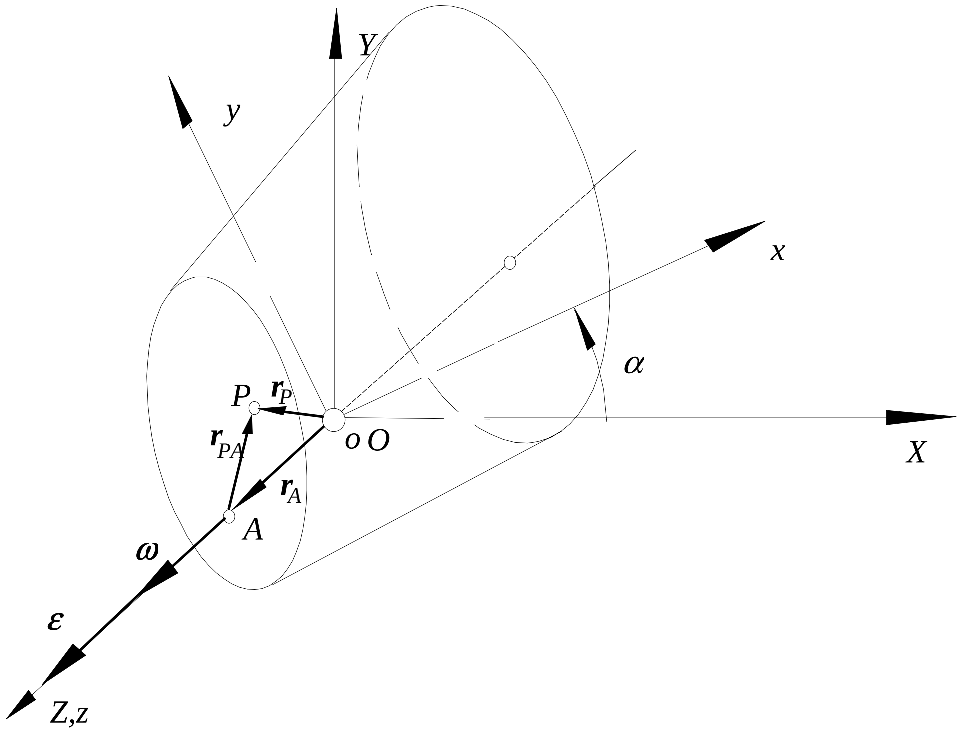


Figure 14

Let us consider an arbitrarily chosen point P that belong to this body. Its absolute position vector $\mathbf{r}_P$ has two components $\mathbf{r}_A$ and $\mathbf{r}_{PA}$ (see Fig. 14).

$$\mathbf{r}_P = \mathbf{r}_A + \mathbf{r}_{PA} \tag{2.27}$$

The vector $\mathbf{r}_A$ determines the plane of motion of the point P whereas the vector $\mathbf{r}_{PA}$ represents distance of the point P from the axis of rotation z.

$$\mathbf{r}_A = \mathbf{k}\cdot z = \mathbf{K}\cdot z; \qquad \mathbf{r}_{PA} = \mathbf{i}\cdot x + \mathbf{j}\cdot y \tag{2.28}$$

Here, x, y and z are the coordinates of the point P with respect to the body system of coordinates xyz.

The absolute velocity of the point P is the time derivative of the absolute position vector $\mathbf{r}_P$.

$$\mathbf{v}_P = \dot{\mathbf{r}}_P = \dot{\mathbf{r}}_A + \dot{\mathbf{r}}_{PA} \tag{2.29}$$

Since coordinates x, y and z are time independent,

$$\begin{aligned} \dot{\mathbf{r}}_A &= 0 \\ \dot{\mathbf{r}}_{PA} &= \mathbf{r}'_{PA} + \boldsymbol{\omega} \times \mathbf{r}_{PA} = \boldsymbol{\omega} \times \mathbf{r}_{PA} \end{aligned} \tag{2.30}$$

Introduction of equations 2.30 into 2.29 yields

$$\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{r}_{PA} \tag{2.31}$$

As one can see from the last relationship, the velocity of the point P does not depend on choice of the plane of motion of the point P. Therefore in the further development, without any harm to generality of the consideration, we will be assuming that the plane of motion coincides with the plane XY (see Fig.15.). In this case $\mathbf{r}_P = \mathbf{r}_{PA}$. Hence

$$\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{r}_P \tag{2.32}$$

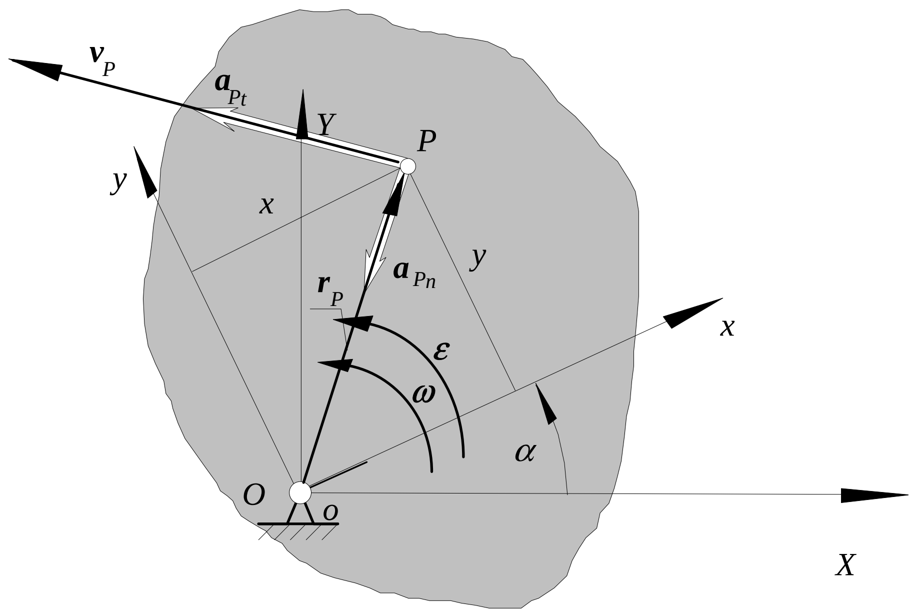


Figure 15

From the relationship 2.32 one can see that the vector of velocity must be perpendicular to both $\boldsymbol{\omega}$ and $\mathbf{r}_P$ as it is shown in Fig. 15. The absolute acceleration can be obtained by differentiation of the absolute velocity vector $\mathbf{v}_P$.

$$\mathbf{a}_P = \dot{\mathbf{v}}_P = \frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r}_P) = \dot{\boldsymbol{\omega}} \times \mathbf{r}_P + \boldsymbol{\omega} \times \mathbf{v}_P = \boldsymbol{\varepsilon} \times \mathbf{r}_P + \boldsymbol{\omega} \times \mathbf{v}_P \tag{2.33}$$

The first term as the cross-product of $\boldsymbol{\varepsilon}$ and $\mathbf{r}_P$ must be perpendicular to both of them and it is called *tangential component of acceleration* $\mathbf{a}_{Pt}$. The second one must be perpendicular to velocity and is called *normal component of acceleration* $\mathbf{a}_{Pn}$. Their magnitudes are as follows

$$\begin{aligned} a_{Pt} &= \varepsilon r_P = \ddot{\alpha} r_P \\ a_{Pn} &= \omega v_P = \omega^2 r_P = \dot{\alpha}^2 r_P \end{aligned} \tag{2.34}$$

General motion of rigid body

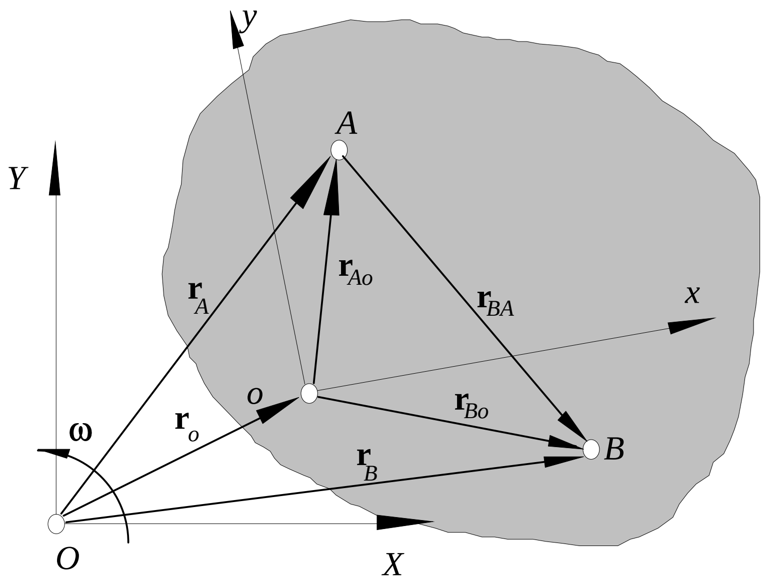


Figure 16

Motion of the translating and rotating system of coordinates xyz shown in Fig. 16 can be uniquely determined by means of two vectors $\mathbf{r}_o$ and $\boldsymbol{\omega}$. The position vector $\mathbf{r}_o$ determines motion of its origin with respect to the inertial system of coordinates XYZ and the vector of angular velocity $\boldsymbol{\omega}$ determines its rotation.

DEFINITION: It is said that a body rigidly attached to the translating and rotating system of coordinates xyz **performs the** *general motion* **and the angular velocity** $\boldsymbol{\omega}$ **of the system of coordinates** xyz **is called** *angular velocity of the rigid body.*

Angular acceleration of the body as the first time derivative of the vector $\boldsymbol{\omega}$ is

$$\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}} = \frac{d}{dt}\left(\mathbf{K}\omega\right) = \mathbf{K}\dot{\omega} \tag{2.35}$$

The absolute velocity of the arbitrarily chosen point A is the first time derivative of its absolute position vector $\mathbf{r}_A$.

$$\mathbf{v}_A = \dot{\mathbf{r}}_A = \frac{d}{dt}(\mathbf{r}_o + \mathbf{r}_{Ao}) = \dot{\mathbf{r}}_o + \dot{\mathbf{r}}_{Ao} \tag{2.36}$$

If the vector $\mathbf{r}_{Ao}$ is resolved along the body system of coordinates xyz, its components are time independent and therefore its derivative is

$$\dot{\mathbf{r}}_{Ao} = \boldsymbol{\omega} \times \mathbf{r}_{Ao} \tag{2.37}$$

Hence

$$\mathbf{v}_A = \dot{\mathbf{r}}_o + \boldsymbol{\omega} \times \mathbf{r}_{Ao} \tag{2.38}$$

In the same manner one can prove that the absolute velocity of the point B is

$$\mathbf{v}_B = \dot{\mathbf{r}}_o + \boldsymbol{\omega} \times \mathbf{r}_{Bo} \tag{2.39}$$

The relative velocity $\mathbf{v}_{BA}$ of the point B with respect to the point A is then

$$\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}_{Bo} - \boldsymbol{\omega} \times \mathbf{r}_{Ao} = \boldsymbol{\omega} \times (\mathbf{r}_{Bo} - \mathbf{r}_{Ao}) = \boldsymbol{\omega} \times \mathbf{r}_{BA} \tag{2.40}$$

Hence the relative velocity between two points that belong to the same rigid body is always perpendicular to the line joining them and its scalar magnitude is

$$v_{BA} = \omega r_{BA} \sin 90^o = \omega r_{BA} \tag{2.41}$$

The sense of this vector, according to the definition of the cross product, can be determined by rotation of the vector $\mathbf{r}_{BA}$ by 90^o in the direction of the angular velocity $\boldsymbol{\omega}$ (see Fig. 17). It can be seen from the equation 2.40 that the absolute velocity of an arbitrarily chosen point B is

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA} = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{BA} \tag{2.42}$$

The absolute acceleration of the point A one can get by differentiation of the vector of absolute velocity

$$\mathbf{a}_A = \dot{\mathbf{v}}_A = \ddot{\mathbf{r}}_o + \frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r}_{Ao}) = \ddot{\mathbf{r}}_o + \boldsymbol{\varepsilon} \times \mathbf{r}_{Ao} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{Ao}) \tag{2.43}$$

Similarly, the absolute acceleration of the point B is

$$\mathbf{a}_B = \dot{\mathbf{v}}_B = \ddot{\mathbf{r}}_o + \frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r}_{Bo}) = \ddot{\mathbf{r}}_o + \boldsymbol{\varepsilon} \times \mathbf{r}_{Bo} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{Bo}) \tag{2.44}$$

The difference between these two above vectors produces the relative acceleration $\mathbf{a}_{BA}$.

$$\begin{aligned} \mathbf{a}_{BA} &= \mathbf{a}_B - \mathbf{a}_A = \boldsymbol{\varepsilon} \times \mathbf{r}_{Bo} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{Bo}) - \boldsymbol{\varepsilon} \times \mathbf{r}_{Ao} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{Ao}) = \\ &= \boldsymbol{\varepsilon} \times (\mathbf{r}_{Bo} - \mathbf{r}_{Ao}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{Bo} - \boldsymbol{\omega} \times \mathbf{r}_{Ao}) = \\ &= \boldsymbol{\varepsilon} \times \mathbf{r}_{BA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r}_{Bo} - \mathbf{r}_{Ao})) = \\ &= \boldsymbol{\varepsilon} \times \mathbf{r}_{BA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{BA}) = \boldsymbol{\varepsilon} \times \mathbf{r}_{BA} + \boldsymbol{\omega} \times \mathbf{v}_{BA} \end{aligned} \tag{2.45}$$

The normal component of the relative acceleration

$$\mathbf{a}_{BAn} = \boldsymbol{\omega} \times \mathbf{v}_{BA} \tag{2.46}$$

and its tangential component

$$\mathbf{a}_{BAt} = \boldsymbol{\varepsilon} \times \mathbf{r}_{BA} \tag{2.47}$$

are shown in Fig. 18.

Eventually we can state that if A and B belong to the same body the relationship between their absolute accelerations is

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA} = \mathbf{a}_A + \boldsymbol{\varepsilon} \times \mathbf{r}_{BA} + \boldsymbol{\omega} \times \mathbf{v}_{BA} \tag{2.48}$$

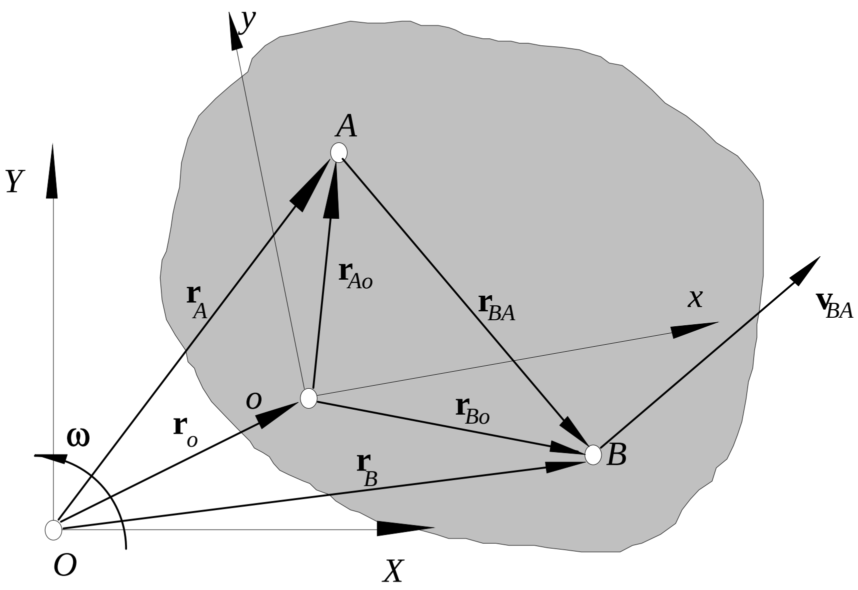


Figure 17

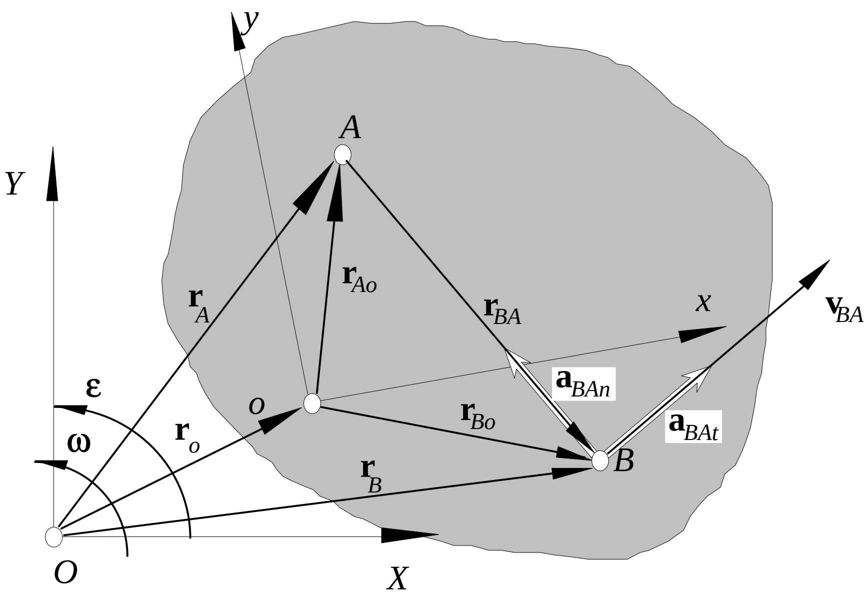


Figure 18

2.1.3 Problems

Problem 18

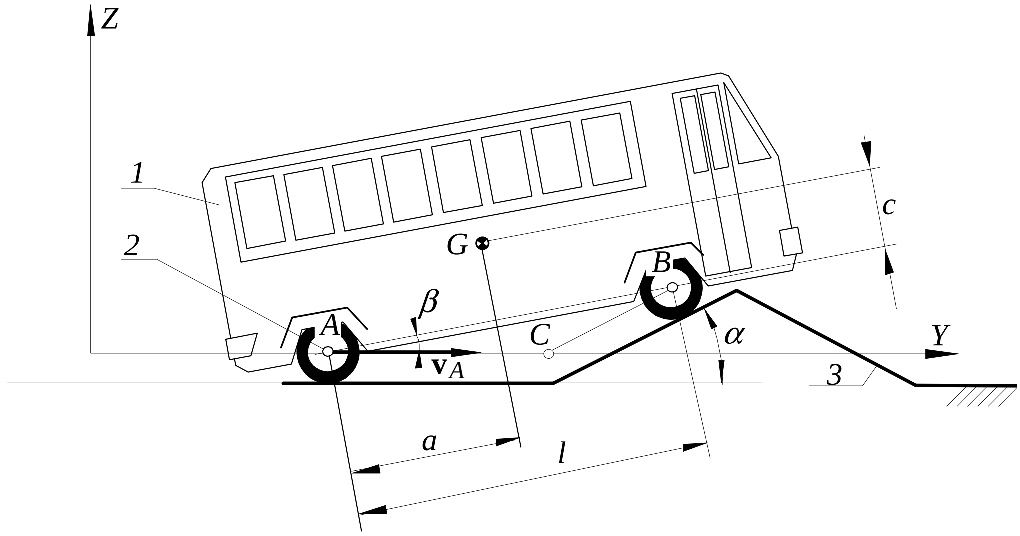


Figure 19

The bus *1* (see Fig. 19) moves in the vertical plane $YZ$of the inertial system of coordinates XYZ. The rear wheels *2* of the bus are driven with the constant linear velocity $\mathbf{v}_A$ The bus is climbing on the hump *3* which is inclined to the horizontal plane at the angle α.

Develop the expression for

1. the absolute angular velocity of the bus
2. the absolute angular acceleration of the bus
3. the absolutethe absolute velocity of the front axle B
4. the absolute linear acceleration of the front axle B
5. the absolutethe absolute velocity of the centre of gravity G
6. the absolute linear acceleration of the centre of gravity G

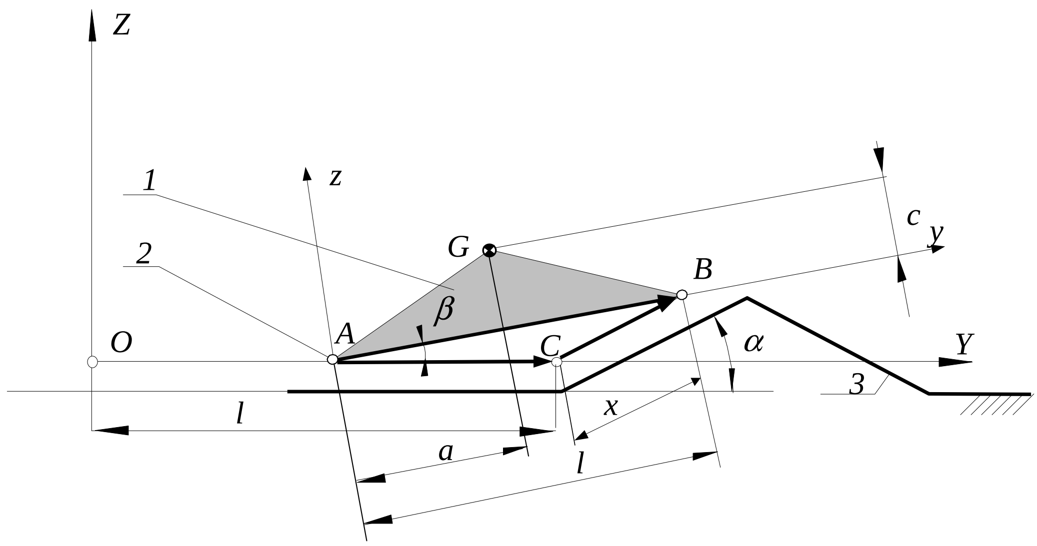


Figure 20

Solution

The origin of the inertial system of coordinates YZ has been chosen to coincide with the axle A when the bus reaches the hump (point B coincides the point C - see Fig. 20). Hence, for any arbitrarily chosen instant of time, the distance OA is

$$AC = l - v_A t \tag{2.49}$$

The system of coordinates xyz is rigidly attached to the bus. Hence the angle β represents the absolute angular displacement of the body 1 system of coordinates. Both the angular displacement and the distance $CB = x$ can be produced by solving the following vector equation

$$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB} \tag{2.50}$$

where

$$\begin{aligned}
\overrightarrow{AB} &= \mathbf{j}l \\
\overrightarrow{AC} &= \mathbf{J}(l - v_A t) \\
\overrightarrow{CB} &= \mathbf{J}x\cos\alpha + \mathbf{K}x\sin\alpha
\end{aligned} \tag{2.51}$$

Hence

$$\mathbf{j}l = \mathbf{J}(l - v_A t) + \mathbf{J}x\cos\alpha + \mathbf{K}x\sin\alpha \tag{2.52}$$

This equation is equivalent to two scalar equations that can be obtained by subsequent multiplication of the equation 2.52 by the unit vectors **J** and **K.**

$$\begin{aligned}
l\cos\beta &= l - v_A t + x\cos\alpha \\
l\sin\beta &= x\sin\alpha
\end{aligned} \tag{2.53}$$

This set has the following solutions

$$\beta = \alpha - \arcsin\left(\frac{l - v_A t}{l}\sin\alpha\right) \tag{2.54}$$

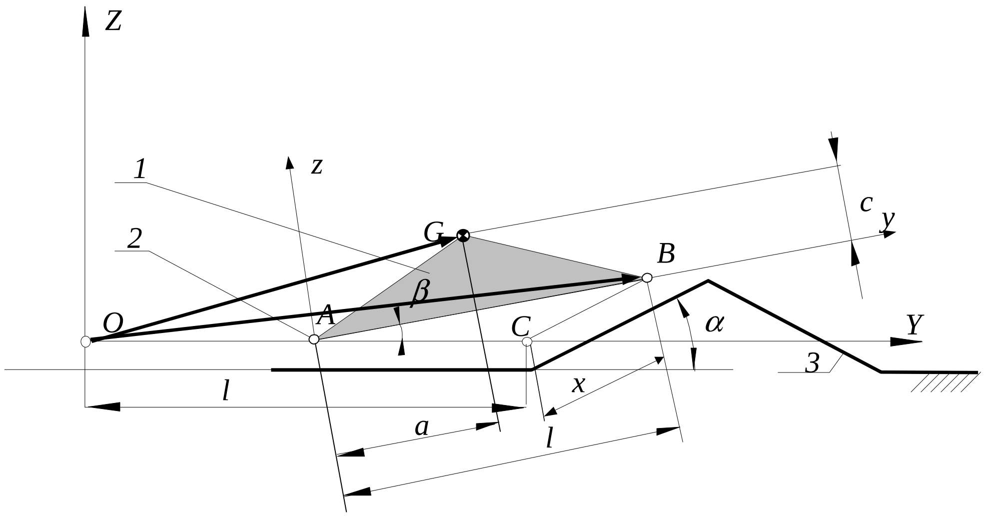


Figure 21

and

$$x = \frac{l \sin\beta}{\sin\alpha} \tag{2.55}$$

Hence the absolute angular velocity is

$$\omega = \mathbf{I}\dot{\beta} \tag{2.56}$$

and the absolute angular acceleration is

$$\varepsilon = \mathbf{I}\ddot{\beta} \tag{2.57}$$

To get the absolute linear velocity of the point B ane has to differentiate the absolute position vector $\mathbf{r}_B$ (see Fig. 21)

$$\mathbf{r}_B = \mathbf{J}(v_A t + l\cos\beta) + \mathbf{K} l \sin\beta \tag{2.58}$$

Hence

$$\mathbf{v}_B = \mathbf{J}(v_A - l\dot{\beta}\sin\beta) + \mathbf{K} l\dot{\beta}\cos\beta \tag{2.59}$$

The derivative of the absolute velocity yields the absolute acceleration

$$\mathbf{a}_B = \mathbf{J}(\,-l\ddot{\beta}\sin\beta - l\dot{\beta}^2\cos\beta) + \mathbf{K}(l\ddot{\beta}\cos\beta - l\dot{\beta}^2\sin\beta) \tag{2.60}$$

Differentiation of the absolute position vector of the point G

$$\mathbf{r}_G = \mathbf{J}(v_A t + a\cos\beta - c\sin\beta) + \mathbf{K}(a\sin\beta + c\cos\beta) \tag{2.61}$$

yields the absolute velocity of the centre of gravity G

$$\mathbf{v}_G = \dot{\mathbf{r}}_G = \mathbf{J}(v_A - a\dot{\beta}\sin\beta - c\dot{\beta}\cos\beta) + \mathbf{K}(a\dot{\beta}\cos\beta - c\dot{\beta}\sin\beta) \tag{2.62}$$

The absolute acceleration of the centre of gravity is

$$\begin{aligned}\mathbf{a}_G \;\;=\;\; & \dot{\mathbf{v}}_G = \mathbf{J}(\,-a\ddot{\beta}\sin\beta - a\dot{\beta}^2\cos\beta - c\ddot{\beta}\cos\beta + c\dot{\beta}^2\sin\beta) + \\ & +\mathbf{K}(a\ddot{\beta}\cos\beta - a\dot{\beta}^2\sin\beta - c\ddot{\beta}\sin\beta - c\dot{\beta}^2\cos\beta)\end{aligned} \tag{2.63}$$

Problem 19

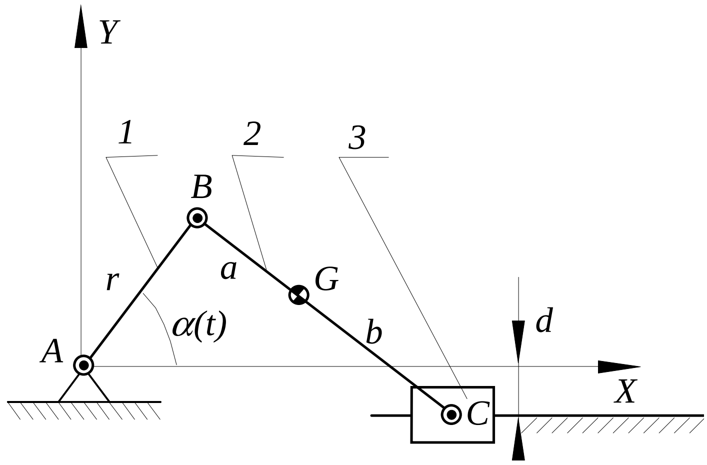


Figure 22

Motion of the crankshaft 1 shown in Fig. 22 is given by its angular position $\alpha(t)$. Produce the expression for the absolute velocity and the absolute acceleration of its piston 3.

Given are $\alpha(t)$, r, a, b, d

Solution

The instantaneous position of the system (X and β) can be obtained by solving the following vector equation (see Fig. 23).

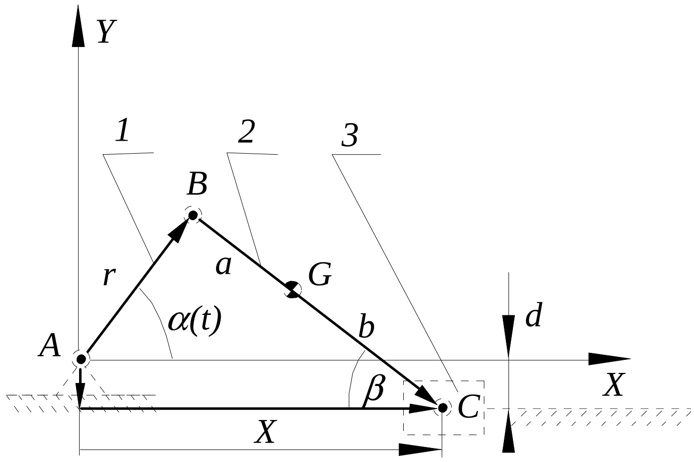


Figure 23

$$\overrightarrow{AB} + \overrightarrow{BC} = \mathbf{J}(-d) + \mathbf{I}X \tag{2.64}$$

or

$$\mathbf{I}r\cos\alpha + \mathbf{J}r\sin\alpha + \mathbf{I}((a+b)\cos\beta) + \mathbf{J}(-(a+b)\sin\beta) = \mathbf{J}(-d) + \mathbf{I}X \tag{2.65}$$

This equation is equivalent to two scalar equations

$$\begin{aligned} r\cos\alpha + (a+b)\cos\beta &= X \\ r\sin\alpha - (a+b)\sin\beta &= -d \end{aligned} \tag{2.66}$$

Hence

$$\sin\beta = \frac{d + r\sin\alpha}{a+b} \tag{2.67}$$

and

$$X = r\cos\alpha + (a+b)\sqrt{1 - \left(\frac{d + r\sin\alpha}{a+b}\right)^2} \tag{2.68}$$

Velocity of the point B is the vector derivative of the position vector $\overrightarrow{AB}$.

$$\mathbf{v}_B = \frac{d}{dt}\left(\mathbf{I}r\cos\alpha + \mathbf{J}r\sin\alpha\right) = \mathbf{I}\left(-r\dot{\alpha}\sin\alpha\right) + \mathbf{J}\left(r\dot{\alpha}\cos\alpha\right) \tag{2.69}$$

According to the relationship between two points of the same body the linear velocity of the point C is

$$\begin{aligned} \mathbf{v}_C &= \mathbf{v}_B + \mathbf{v}_{CB} = \mathbf{v}_B + \boldsymbol{\omega}_2 \times \overrightarrow{BC} = \\ &= \mathbf{I}\left(-r\dot{\alpha}\sin\alpha\right) + \mathbf{J}\left(r\dot{\alpha}\cos\alpha\right) + \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 0 & 0 & \omega_2 \\ (a+b)\cos\beta & -(a+b)\sin\beta & 0 \end{vmatrix} = \\ &= \mathbf{I}\left(-r\dot{\alpha}\sin\alpha + (a+b)\omega_2\sin\beta\right) + \mathbf{J}\left(r\dot{\alpha}\cos\alpha + (a+b)\omega_2\cos\beta\right) \end{aligned} \tag{2.70}$$

According to the constraints shown in Fig. 22, the component of the velocity $\mathbf{v}_C$ along axis Y must be equal to zero.

$$r\dot{\alpha}\cos\alpha + (a+b)\omega_2\cos\beta = 0 \tag{2.71}$$

Hence, the angular velocity of the link 2 is

$$\omega_2 = -\frac{r\dot{\alpha}\cos\alpha}{(a+b)\cos\beta} \tag{2.72}$$

The same result one can get by differentiation of the absolute angular displacement $\gamma = 360^0 - \beta)$ (see Fig. 24)

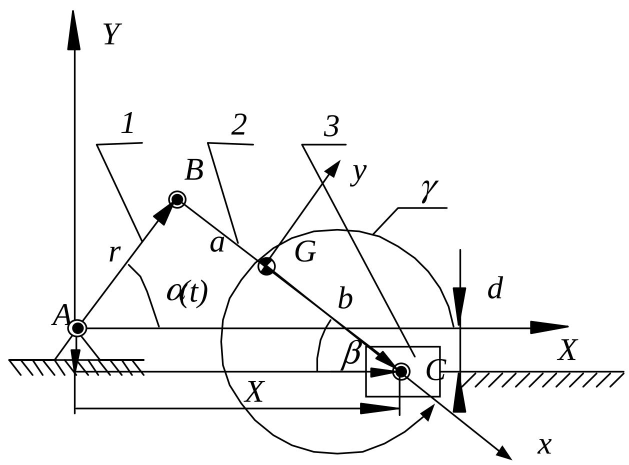


Figure 24

$$\omega_2 = -\dot{\beta} = -\frac{d}{dt}\arcsin\left(\frac{d + r\sin\alpha}{a+b}\right) = -\frac{r\dot{\alpha}\cos\alpha}{(a+b)\cos\beta} \tag{2.73}$$

The velocity of the point C, according to 2.70 is

$$\mathbf{v}_C = \mathbf{I}\left(-r\dot{\alpha}\sin\alpha + (a+b)\omega_2\sin\beta\right) \tag{2.74}$$

Differentiation of the above expression with respect to time yields the wanted acceleration of the point C.

$$\mathbf{a}_C = \mathbf{I}\left(-r\ddot{\alpha}\sin\alpha - r\dot{\alpha}^2\cos\alpha - r\ddot{\alpha}\cos\alpha\tan\beta + r\dot{\alpha}^2\sin\alpha\tan\beta - r\dot{\alpha}\dot{\beta}\frac{1}{\cos^2\beta}\right) \tag{2.75}$$

Problem 20

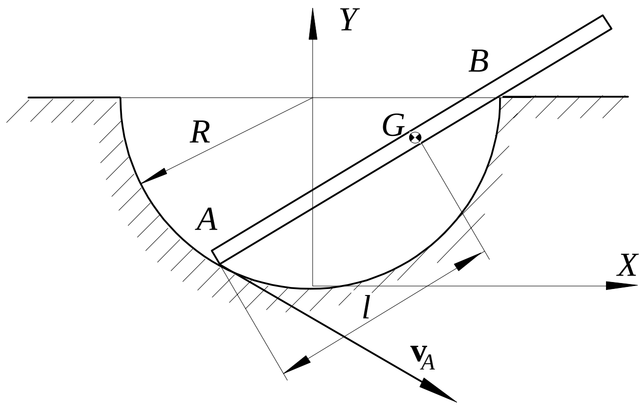


Figure 25

The end A of the rod shown in Fig. 25 slides over the cylindrical surface with the constant velocity v_A. Derive the expression for the angular velocity and acceleration of the rod and the linear velocity and acceleration of its centre of gravity G.

Solution

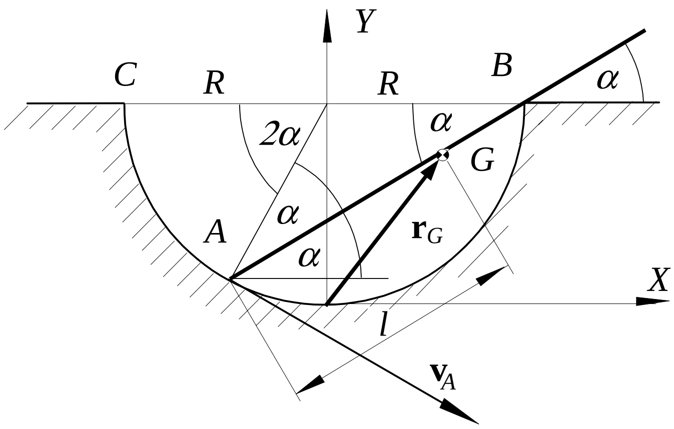


Figure 26

Let us denote the absolute angular displacement of the rod by α as shown in the Fig. 26. It can be seen from this drawing that

$$\overset{\frown}{CA} = 2R\alpha \tag{2.76}$$

Since the point A moves with the constant velocity $\mathbf{v}_A$

$$\overset{\frown}{CA} = v_A t \tag{2.77}$$

Hence,

$$\alpha = \frac{v_A t}{2R} \tag{2.78}$$

Differentiation of the above relationship yields the absolute angular velocity

$$\omega = \dot{\alpha} = \frac{v_A}{2R} \tag{2.79}$$

The absolute angular acceleration is

$$\varepsilon = \dot{\omega} = 0 \tag{2.80}$$

The absolute linear velocity of the point G can be obtained by differentiation of the absolute position vector $\mathbf{r}_G$.

$$\mathbf{r}_G = \mathbf{I}(l\cos\alpha - R\cos 2\alpha) + \mathbf{J}(l\sin\alpha + (R - R\sin 2\alpha)) \tag{2.81}$$

Hence the absolute velocity of the point G is

$$\mathbf{v}_G = \dot{\mathbf{r}}_G = \mathbf{I}(-l\dot{\alpha}\sin\alpha + R2\dot{\alpha}\sin 2\alpha) + \mathbf{J}(l\dot{\alpha}\cos\alpha - R2\dot{\alpha}\cos 2\alpha)) \tag{2.82}$$

the absolute acceleration of the point G is

$$\begin{aligned}\mathbf{a}_G \quad = \quad & \dot{\mathbf{v}}_G = \mathbf{I}(-l\ddot{\alpha}\sin\alpha - l\dot{\alpha}^2\cos\alpha + 2R\ddot{\alpha}\sin 2\alpha + 4R\dot{\alpha}^2\cos 2\alpha) \\ & +\mathbf{J}(l\ddot{\alpha}\cos\alpha - l\dot{\alpha}^2\sin\alpha - 2R\ddot{\alpha}\cos 2\alpha + 4R\dot{\alpha}^2\sin 2\alpha))\end{aligned} \tag{2.83}$$

The absolute velocity of the point B which belong to the rod can be obtained by differentiation of its absolute position vector or from the following relationship between two points which belong to the same body.

$$
\begin{aligned}
\mathbf{v}_B &= \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{BA} \\
&= \mathbf{I} v_A \sin 2\alpha + \mathbf{J}(-v_A \cos 2\alpha) + \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 0 & 0 & \dot{\alpha} \\ 2R\cos\alpha\cos\alpha & 2R\cos\alpha\sin\alpha & 0 \end{vmatrix} \\
&= \mathbf{I}(v_A \sin 2\alpha - 2R\dot{\alpha}\cos\alpha\sin\alpha) + \mathbf{J}(-v_A\cos 2\alpha + 2R\dot{\alpha}\cos\alpha\cos\alpha) \\
&= \mathbf{I}(v_A \sin 2\alpha - v_A\cos\alpha\sin\alpha) + \mathbf{J}(-v_A\cos 2\alpha + v_A\cos\alpha\cos\alpha) = \\
&= \mathbf{I} v_A \cos\alpha\sin\alpha + \mathbf{J} v_A \sin^2\alpha
\end{aligned} \tag{2.84}
$$

As one could predict, the absolute velocity of the point B possesses always direction parallel to the position of the rod

$$\frac{v_{BY}}{v_{BX}} = \frac{v_A \sin^2\alpha}{v_A \cos\alpha\sin\alpha} = \tan\alpha \tag{2.85}$$

and its scalar magnitude

$$v_B = v_A\sqrt{(\cos\alpha\sin\alpha)^2 + \sin^4\alpha} = v_A \sin\alpha \tag{2.86}$$

is equal to the component of the velocity of the point A along the rod

Problem 21

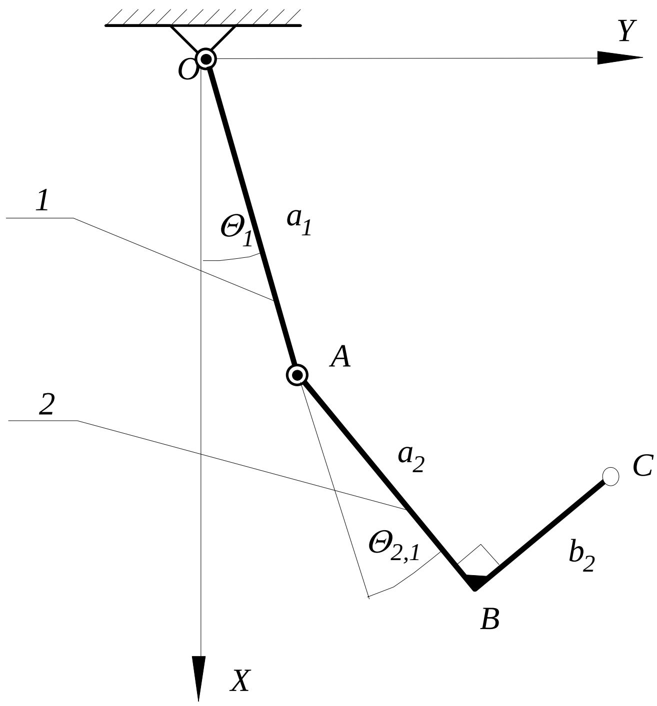


Figure 27

The two elements 1 and 2 are joint together to form the double pendulum shown in Fig. 27. Motion of the element 1 is given by the angular displacement $\Theta_1(t)$. The relative motion of the element 2 with respect to the element 1 is determined by the angular displacement $\Theta_{2,1}(t)$.

Produce:

1. the expression for the absolute angular velocity and acceleration of the link 2

Answer:

$\boldsymbol{\omega}_2 = \mathbf{K}(\dot\Theta_1 + \dot\Theta_{21})$ $\quad\quad$ $\boldsymbol{\varepsilon}_2 = \mathbf{K}(\ddot\Theta_1 + \ddot\Theta_{21})$

2. the linear velocity and acceleration of the point C

Answer:

$\mathbf{v}_C = \mathbf{I}(-a_1\dot\Theta_1 \sin\Theta_1 - a_2(\dot\Theta_1 + \dot\Theta_{21})\sin(\Theta_1 + \Theta_{21}) - b_2(\dot\Theta_1 + \dot\Theta_{21})\cos(\Theta_1 + \Theta_{21}))+$
$+\mathbf{J}(a_1\dot\Theta_1 \cos\Theta_1 + a_2(\dot\Theta_1 + \dot\Theta_{21})\cos(\Theta_1 + \Theta_{21}) - b_2(\dot\Theta_1 + \dot\Theta_{21})\sin(\Theta_1 + \Theta_{21}))$
$\mathbf{a}_C = \mathbf{I}(-a_1\ddot\Theta_1 \sin\Theta_1 - a_1\dot\Theta_1^2 \cos\Theta_1 - a_2(\ddot\Theta_1 + \ddot\Theta_{21})\sin(\Theta_1 + \Theta_{21}) - a_2(\dot\Theta_1 + \dot\Theta_{21})^2\cos(\Theta_1 +$
$\Theta_{21}) - b_2(\ddot\Theta_1 + \ddot\Theta_{21})\cos(\Theta_1 + \Theta_{21}) + b_2(\dot\Theta_1 + \dot\Theta_{21})^2\sin(\Theta_1 + \Theta_{21}))+$
$+\mathbf{J}(a_1\ddot\Theta_1 \cos\Theta_1 - a_1\dot\Theta_1^2 \sin\Theta_1 + a_2(\ddot\Theta_1 + \ddot\Theta_{21})\cos(\Theta_1 + \Theta_{21}) - a_2(\dot\Theta_1 + \dot\Theta_{21})^2\sin(\Theta_1 +$
$\Theta_{21}) - b_2(\ddot\Theta_1 + \ddot\Theta_{21})\sin(\Theta_1 + \Theta_{21}) - b_2(\dot\Theta_1 + \dot\Theta_{21})^2\cos(\Theta_1 + \Theta_{21}))$

Problem 22

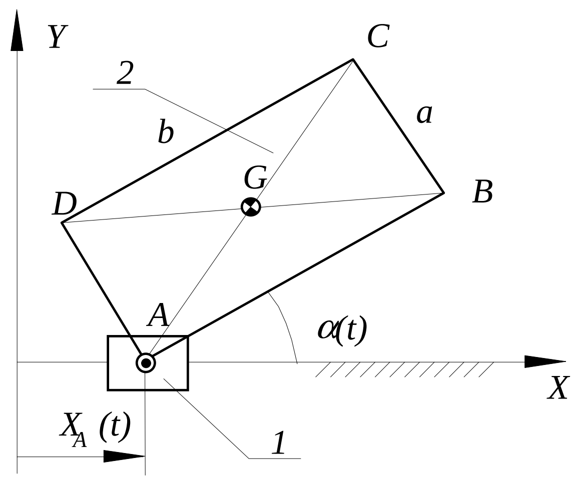


Figure 28

The link 1 of the system shown in Fig. 28 moves along axis X of the inertial system of coordinates XY and its motion is given by the function of time $X_A(t)$. The link 2 of the rectangular shape is hinged to the link *1* at the point A. Its relative motion with respect to the link 1 is given by the function of time $\alpha(t)$.

Produce:

1. the expression for the absolute angular acceleration of the link 2

Answer:

$\boldsymbol{\omega}_2 = \mathbf{K}\dot{\alpha} \qquad \boldsymbol{\varepsilon}_2 = \mathbf{K}\ddot{\alpha}$

2. the expression for the absolute linear velocity and linear acceleration of the point G.

Answer:

$\mathbf{v}_G = \mathbf{I}(\dot{X}_A - \frac{b}{2}\dot{\alpha}\sin\alpha - \frac{a}{2}\dot{\alpha}\cos\alpha) + \mathbf{J}(\frac{b}{2}\dot{\alpha}\cos\alpha - \frac{a}{2}\dot{\alpha}\sin\alpha)$

$\mathbf{a}_G = \mathbf{I}(\ddot{X}_A - \frac{b}{2}\ddot{\alpha}\sin\alpha - \frac{b}{2}\dot{\alpha}^2\cos\alpha - \frac{a}{2}\ddot{\alpha}\cos\alpha + \frac{a}{2}\dot{\alpha}^2\sin\alpha) + \mathbf{J}(\frac{b}{2}\ddot{\alpha}\cos\alpha - \frac{b}{2}\dot{\alpha}^2\sin\alpha - \frac{a}{2}\ddot{\alpha}\sin\alpha - \frac{a}{2}\dot{\alpha}^2\cos\alpha)$

Problem 23

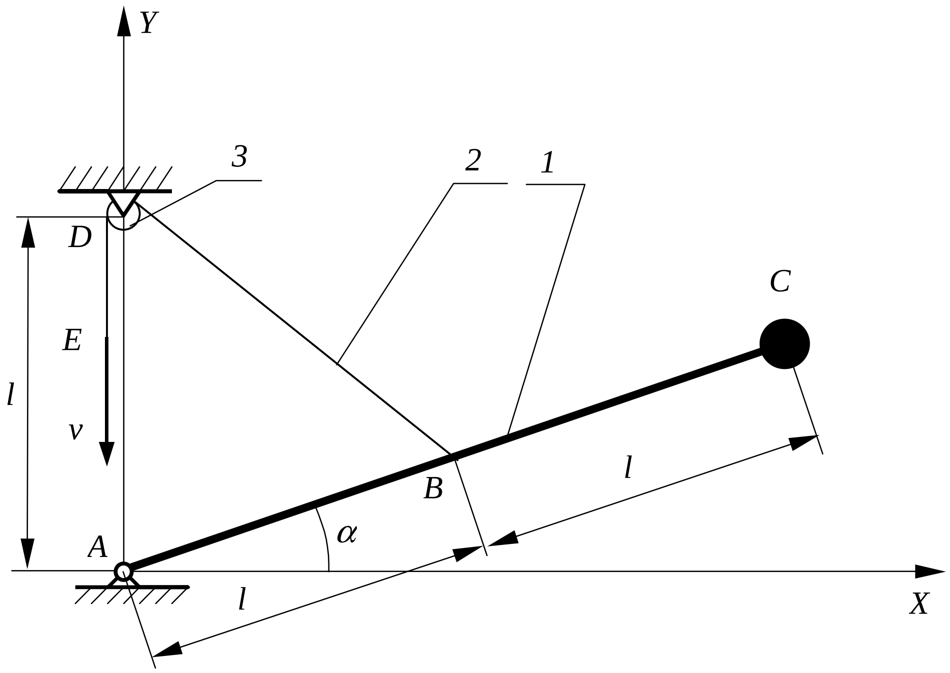


Figure 29

The link 1 of the system shown in Fig. 29 can be treated as a rigid body. This link is being raised by means of the rope 2, the pulley 3. The radius of the pulley is negligible. The point E of the rope moves with the constant velocity v.

Produce:

1. the angular displacement $\alpha$of the link 1 as a function of time
Answer:
$\alpha = 90^o - 2\arcsin\frac{\sqrt{2}l-vt}{2l}$
2. the angular velocity of the link 1 as a function of time
Answer:
$\dot{\alpha} = \frac{v}{l\cos\left(45^o-\frac{\alpha}{2}\right)} = \frac{1}{l}\frac{v}{\sqrt{\frac{1}{4l^2}\left(2l^2-t^2v^2+2ltv\sqrt{2}\right)}}$
3. the absolute velocity of the point C
Answer:
$v_C = 2l\dot{\alpha}$

2.2 PLANE KINETICS OF A RIGID BODY

2.2.1 Kinetic energy

Rotation of rigid body

Let us consider a rigid body that rotates with an angular velocity $\boldsymbol{\omega}$ about the axis Z of the absolute system of coordinates XYZ (see Fig. 30).The kinetic energy of a

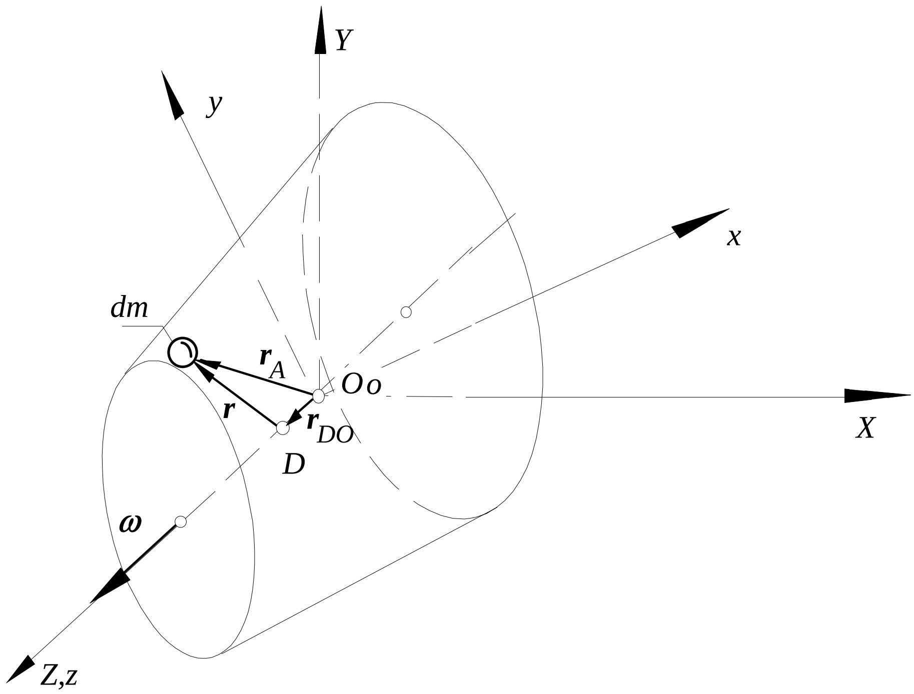


Figure 30

particle of mass dm is

$$dT = \frac{1}{2}v^2 dm \tag{2.87}$$

where v represents the absolute velocity of the particle. As it has been shown in the previous chapter, magnitude of the absolute velocity v is

$$v = |\dot{\mathbf{r}}_A| = |\boldsymbol{\omega} \times \mathbf{r}| \tag{2.88}$$

where $\boldsymbol{\omega}$ is the absolute angular velocity of the body and $\mathbf{r}$ is the vector that represents the distance of the particle from the axis of rotation. Since these two vectors are perpendicular to each other and $dm = \rho dV$

$$dT = \frac{1}{2}\omega^2 r^2 \rho dV \tag{2.89}$$

Hence, the total kinetic energy is the following volume integral

$$T = \frac{1}{2}\omega^2 \left(\iiint_V r^2 \rho dV \right) \tag{2.90}$$

The expression in brackets is called (mass) moment of inertia and is usually denoted by I. Since in our case the body rotates about axis z through the point O we will denote it by I_{zO}. As a matter of convenience the above volume integral will be denoted by $\int\limits_m$.

$$I_{zO} = \iiint_V r^2 \rho dV = \int\limits_m r^2 dm \tag{2.91}$$

General motion.

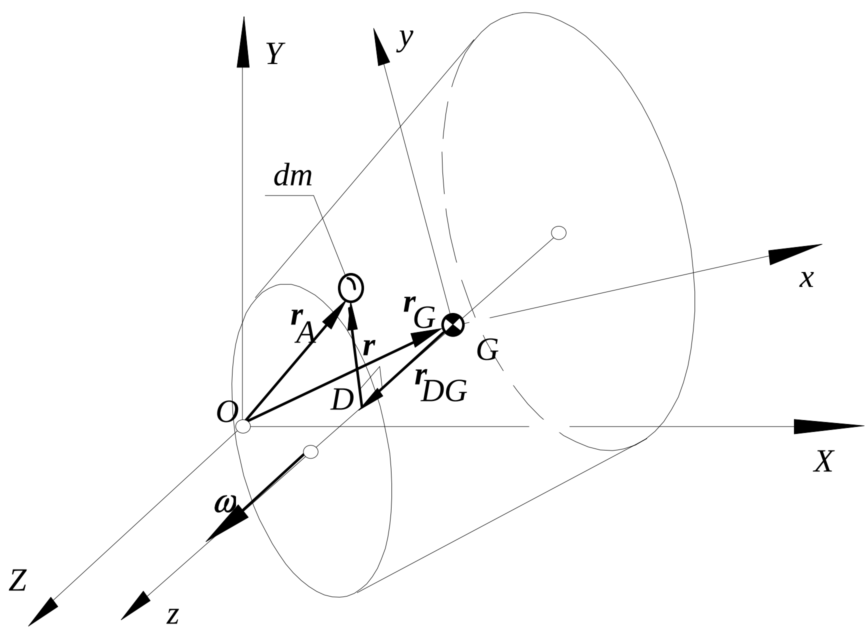


Figure 31

Kinetic energy of a particle dm of a rigid body (see Fig. 31) is

$$dT = \frac{1}{2}\dot{\mathbf{r}}_A^2 dm \tag{2.92}$$

where $\mathbf{r}_A$ is the absolute position vector of the particle dm. If the body performs the general motion, the position vector $\mathbf{r}_A$ may be defined as follows

$$\mathbf{r}_A = \mathbf{r}_G + \mathbf{r}_{D,G} + \mathbf{r} \tag{2.93}$$

The vector $\mathbf{r}_G$ determines position of the centre of gravity G with respect to the inertial space XYZ. The vector $\mathbf{r}_{D,G} = \mathbf{K}r_{D,G}$ determines the plane of motion of the particle dm with respect to the centre of gravity. Since the magnitude of $\mathbf{r}_{D,G}$ does not depend on time and $\mathbf{K}$ is unit vector associated with the absolute system of coordinates, its derivative is equal to 0. Hence, differentiation of the above vector yields

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_G + \dot{\mathbf{r}}$$

Introducing Eq. 2.93 into Eq. 2.92 one can obtain that

$$dT = \frac{1}{2}(\dot{\mathbf{r}}_G + \dot{\mathbf{r}})^2 dm = \frac{1}{2}(\dot{\mathbf{r}}_G^2 dm + \dot{\mathbf{r}}^2 dm + 2\dot{\mathbf{r}}_G \cdot \dot{\mathbf{r}} dm) \tag{2.94}$$

Integration over the entire body yields the total kinetic energy in the fallowing form

$$T = \frac{1}{2}(\dot{\mathbf{r}}_G^2 \int\limits_m dm + \int\limits_m \dot{\mathbf{r}}^2 dm + 2\dot{\mathbf{r}}_G \cdot \int\limits_m \dot{\mathbf{r}} dm) \tag{2.95}$$

It is easy to see that the first integral yields the entire mass of the body

$$\int\limits_m dm = m \tag{2.96}$$

Let us consider the second integral

$$\int\limits_m \dot{\mathbf{r}}^2 dm = \int\limits_m (\boldsymbol{\omega} \times \mathbf{r})^2 dm = \omega^2 \int\limits_m r^2 dm = \omega^2 I_{zG} \tag{2.97}$$

In the last expression I_{zG} stands for the (mass) moment of inertia of the body about axis z through the centre of gravity G.

Let us consider the last integral

$$\int\limits_m \dot{\mathbf{r}} dm = \int\limits_m \boldsymbol{\omega} \times \mathbf{r} dm = \boldsymbol{\omega} \times \int\limits_m \mathbf{r} dm \tag{2.98}$$

On the other hand, the distance between the centre of gravity and the axis z is determine by the following formula

$$\mathbf{r}_{o,G} = \frac{1}{m} \int\limits_m \mathbf{r} dm \tag{2.99}$$

Since in the case consider the system of coordinates was chosen through the centre of gravity G, $\mathbf{r}_{o,G}$ is equal to zero. It follows that the integral $\int\limits_m \mathbf{r} dm$ must be equal to zero too. Hence

$$\int\limits_m \dot{\mathbf{r}} dm = 0 \tag{2.100}$$

Implementation of Eq's. 2.96, 2.97 and 2.100 into Eq. 2.95 gives the following formula for kinetic energy.

$$T = \frac{1}{2} v_G^2 m + \frac{1}{2} \omega^2 \cdot I_{zG} \tag{2.101}$$

where:

v_G is the absolute velocity of the centre of gravity of the body
ω is the absolute angular velocity of the body
m is mass of the body

I_{zG} is mass moment of inertia about axis through centre of gravity and perpendicular to the plane of motion

The last formula permits to formulate the following statement.

STATEMENT: Kinetic energy of a rigid body is equal to a sum of its energy in the translational motion with velocity of its centre of gravity ($\frac{1}{2}v_G^2 m$ *energy of translation*) **and energy in the rotational motion about axis through its centre of gravity** ($\frac{1}{2}\omega^2 \cdot I_{zG}$ *energy of rotation*).

2.2.2 ***Properties of the mass moment of inertia***

The introduced definition of the mass moment of inertia about axis z

$$I_z = \int_m r^2 dm \tag{2.102}$$

allows us to calculate it for bodies having a simple geometrical shape like a cylinder, sphere, rectangular block etc. As an example let as develop formula for the moment of inertia of a cylinder of the radius R, the length L and the density ϱ. (see Fig. 32).According to the introduced definition, the mass moment of inertia of the cylinder

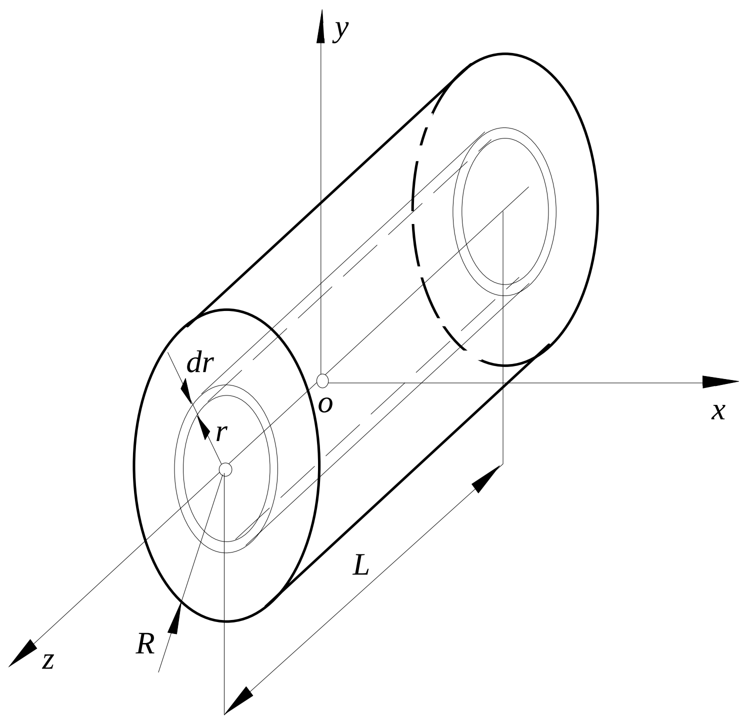


Figure 32

of the radius r and the thickness of its wall dr is

$$dI_z = r^2 dm = r^2 dAL\varrho = r^2 2\pi r dr L\varrho = 2\pi L\varrho r^3 dr$$

Hence the moment of inertia of the whole cylinder is

$$I_z = \int_0^R 2\pi L\varrho r^3 dr = 2\pi L\varrho \int_0^R r^3 dr = 2\pi L\varrho \left|\frac{r^4}{4}\right|_0^R = \pi L\varrho \frac{R^4}{2} = \pi R^2 L\varrho \frac{R^2}{2} = m\frac{R^2}{2}$$

where m is total mass of the cylinder

In a similar manner the formulae for different solids can be derived. Results of such derivations are collected in the appendix B.

For bodies having more complicated shape (see Fig. 33) a division into small elements has to be carried out. Then, each element can be considered as a particle

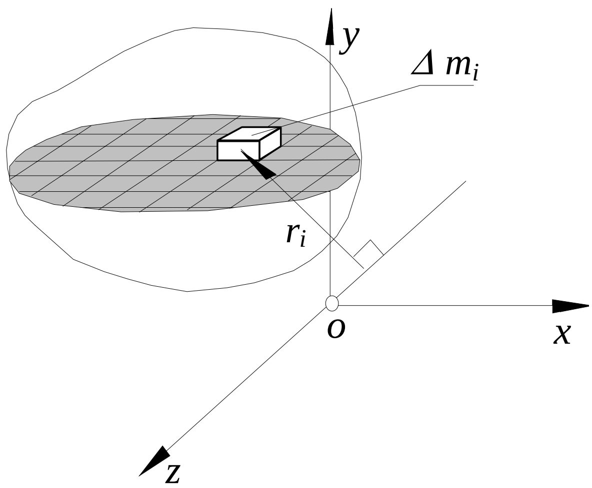


Figure 33

and the integration may be replaced by summation.

$$I_z \approx \sum_{i=1}^{N} r_i^2 \Delta m_i \tag{2.103}$$

Parallel axis theorem

Let us assume, that the moment of inertia I_z of a body is known about the axis z of the xyz system of coordinates (Fig. 34). Let a, b be coordinates of the centre of gravity G of the body considered. Let x_G, y_G, z_G be the body system of coordinates parallel to xyz having its origin at G.Moment of inertia of the body about the axis

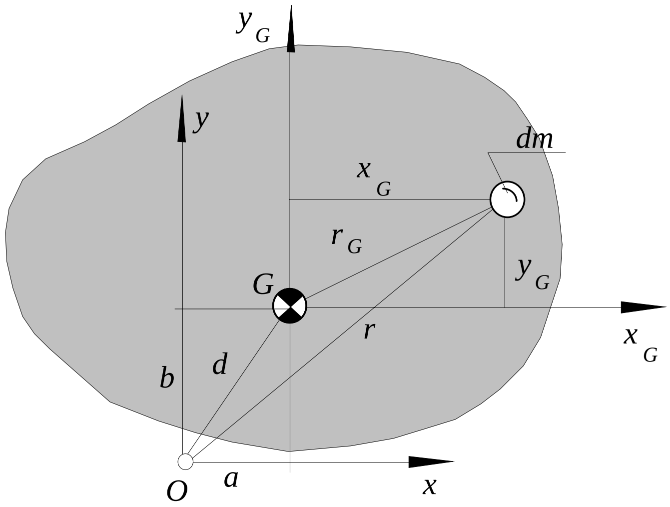


Figure 34

z, according to the previously introduced definition, is

$$
\begin{aligned}
I_z &= \int_m r^2 dm = \int_m ((a + x_G)^2 + (b + y_G)^2) dm = \\
&= \int_m (a^2 + x_G^2 + 2ax_G + b^2 + y_G^2 + 2by_G) dm \\
&= \int_m (x_G^2 + y_G^2) dm + \int_m (a^2 + b^2) dm + \int_m 2ax_G dm + \int_m 2by_G dm \\
&= \int_m r_G^2 dm + \int_m d^2 dm + \int_m 2ax_G dm + \int_m 2by_G dm \\
&= I_{z_G} + d^2 \int_m dm + 2a \int_m x_G dm + 2b \int_m y_G dm
\end{aligned} \tag{2.104}
$$

But,

$$
\frac{1}{m} \int_m x_G dm, \qquad \frac{1}{m} \int_m y_G dm \tag{2.105}
$$

represents components of the distance between origin G and the centre of gravity G, which is actually 0. Hence,

$$
I_z = I_{z_G} + d^2 m \tag{2.106}
$$

The last formula is known as the *parallel axes theorem.*

2.2.3 Equations of motion

General motion of rigid body

Let us consider a rigid body that performs the general motion with respect to the inertial space XYZ (see Fig 35).

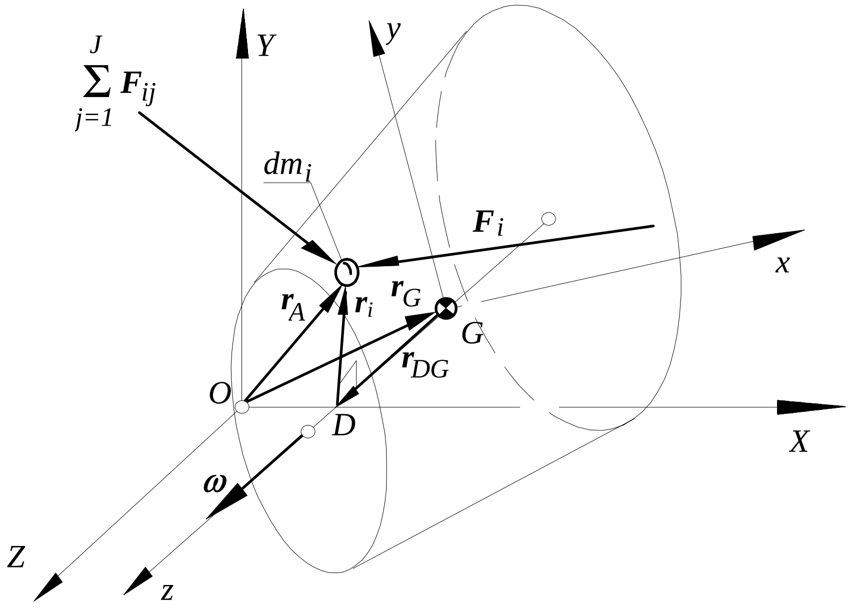


Figure 35

This motion is uniquely determined by the position vector $\mathbf{r}_G$ and the angular velocity $\boldsymbol{\omega}$. Let us split the rigid body into individual particles the body is made of. Motion of one particle dm_i is governed by the Newton's law. It says that the product of mass of the particle and the vector of its absolute acceleration is equal to the vector of the resultant force acting on the particle. The forces that act on the particle can be divided into two categories external and internal. The external forces are due to the interaction of this particle with the external sources such as the Earth (gravity forces) or another elements that do not belong to the body considered. Let us denote the resultant force due to the external forces by $\mathbf{F}_i$. The internal forces are due to the interaction of the particle i with all the other particles the body is made of. The resultant of all the internal forces is denoted here by $\sum_{j=1}^{J} \mathbf{F}_{ij}$. Hence, application of the Newton's second law to the particle i yields

$$dm_i \ddot{\mathbf{r}}_A = \mathbf{F}_i + \sum_{j=1}^{J} \mathbf{F}_{ij} \tag{2.107}$$

In the above equation $\mathbf{r}_A$ stands for the absolute position vector. According to the Fig. 35 it can be expressed as follows

$$\mathbf{r}_A = \mathbf{r}_G + \mathbf{r}_{DG} + \mathbf{r}_i \tag{2.108}$$

Two subsequent differentiations with respect to time produce the wanted absolute acceleration of the particle i.

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_G + \boldsymbol{\omega} \times \mathbf{r}_i \tag{2.109}$$

$$\ddot{\mathbf{r}}_A = \ddot{\mathbf{r}}_G + \dot{\boldsymbol{\omega}} \times \mathbf{r}_i + \boldsymbol{\omega} \times \dot{\mathbf{r}}_i \tag{2.110}$$

Introduction of the above expression into equation 2.107 yields

$$dm_i\ddot{\mathbf{r}}_G + dm_i\dot{\boldsymbol{\omega}} \times \mathbf{r}_i + dm_i\boldsymbol{\omega} \times \dot{\mathbf{r}}_i = \mathbf{F}_i + \sum_{j=1}^{J} \mathbf{F}_{ij} \tag{2.111}$$

It is possible to create such an equation for each particle the body is made of. Then, we can add these equation together getting

$$\sum_{i=1}^{I} dm_i\ddot{\mathbf{r}}_G + \sum_{i=1}^{I} dm_i\dot{\boldsymbol{\omega}} \times \mathbf{r} + \sum_{i=1}^{I} dm_i\boldsymbol{\omega} \times \dot{\mathbf{r}} = \sum_{i=1}^{I} \mathbf{F}_i + \sum_{i=1}^{I}\sum_{j=1}^{J} \mathbf{F}_{ij} \tag{2.112}$$

Let us consider each term of equation 2.112 separately.

$$\sum_{i=1}^{I} dm_i\ddot{\mathbf{r}}_G = \ddot{\mathbf{r}}_G \sum_{i=1}^{I} dm_i = m\ddot{\mathbf{r}}_G \tag{2.113}$$

$$\sum_{i}^{I=1} dm_i\dot{\boldsymbol{\omega}} \times \mathbf{r}_i = \dot{\boldsymbol{\omega}} \times \sum_{i}^{I=1} dm_i\mathbf{r}_i \tag{2.114}$$

The expression $\frac{1}{m}\sum_{i}^{I=1} dm_i\mathbf{r}_i$ represents offset of the centre of gravity of the body from the axis z. In the case considered this distance is equal to 0. Hence

$$\sum_{i}^{I=1} dm_i\dot{\boldsymbol{\omega}} \times \mathbf{r}_i = \mathbf{0} \tag{2.115}$$

For the same reason the next term is equal to zero too.

$$\sum_{i=1}^{I} dm_i\boldsymbol{\omega} \times \dot{\mathbf{r}}_i = \boldsymbol{\omega} \times \sum_{i=1}^{I} dm_i\dot{\mathbf{r}}_i = \boldsymbol{\omega} \times \sum_{i=1}^{I} dm_i\boldsymbol{\omega} \times \mathbf{r}_i = \boldsymbol{\omega} \times \boldsymbol{\omega} \times \sum_{i=1}^{I} dm_i\mathbf{r}_i = \mathbf{0} \tag{2.116}$$

$\sum_{i=1}^{I} \mathbf{F}_i$ represents the resultant of all external forces acting on the rigid body and it will be denoted here by $\mathbf{F}$.

$$\sum_{i=1}^{I} \mathbf{F}_i = \mathbf{F} \tag{2.117}$$

Since the particle i can not interacts with itself,.F_{ii} in the last term of equation 2.112 must be equal to zero. Therefore, the double sum $\sum_{i=1}^{I}\sum_{j=1}^{J} \mathbf{F}_{ij}$ is assembled of the following elements

$$\mathbf{F}_{ij} + \mathbf{F}_{ji}$$

which, according to the Newton's third law ($\mathbf{F}_{ij} = -\mathbf{F}_{ji}$), are equal to zero. Hence

$$\sum_{i=1}^{I}\sum_{j=1}^{J} \mathbf{F}_{ij} = 0 \tag{2.118}$$

Introducing equations from 2.113 to 2.118 into equation 2.112 one can obtain

$$m\ddot{\mathbf{r}}_G = \mathbf{F} \tag{2.119}$$

Another independent equation we can get by multiplying left and right hand side of equation 2.132 by the vector $\mathbf{r}$.

$$\mathbf{r} \times (dm_i\ddot{\mathbf{r}}_G + dm_i\dot{\boldsymbol{\omega}} \times \mathbf{r}_i + dm_i\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) = \mathbf{r}_i \times (\mathbf{F}_i + \sum_{j=1}^{J} \mathbf{F}_{ij}) \tag{2.120}$$

or

$$dm_i\mathbf{r}_i \times \ddot{\mathbf{r}}_G + dm_i\mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) + dm_i\mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) = \mathbf{r}_i \times \mathbf{F}_i + \mathbf{r}_i \times \sum_{j=1}^{J} \mathbf{F}_{ij} \tag{2.121}$$

Let us consider the triple cross-product existing in the above equation.

$$\mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) = \dot{\boldsymbol{\omega}} \cdot (\mathbf{r}_i \cdot \mathbf{r}_i) - \mathbf{r}_i \cdot (\mathbf{r}_i \cdot \dot{\boldsymbol{\omega}}) \tag{2.122}$$

Taking into account that $\dot{\boldsymbol{\omega}} = \mathbf{k}\varepsilon$, $\mathbf{r}_i \cdot \mathbf{r}_i = r_i^2$, and since the vectors $\mathbf{r}_i$ and $\dot{\boldsymbol{\omega}}$ are perpendicularthat $\mathbf{r}_i \cdot \dot{\boldsymbol{\omega}} = 0$ the above triple cross-product is

$$\mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) = \mathbf{k}\varepsilon r_i^2 \tag{2.123}$$

Similarly

$$\begin{aligned} \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) &= \boldsymbol{\omega} \cdot (\dot{\mathbf{r}}_i \cdot \mathbf{r}_i) - \dot{\mathbf{r}}_i \cdot (\mathbf{r}_i \cdot \boldsymbol{\omega}) = \boldsymbol{\omega} \cdot (\dot{\mathbf{r}}_i \cdot \mathbf{r}_i) = \boldsymbol{\omega} \cdot ((\boldsymbol{\omega} \times \mathbf{r}_i) \cdot \mathbf{r}_i)) \\ &= \boldsymbol{\omega} \cdot ((\mathbf{r}_i \times \mathbf{r}_i) \cdot \boldsymbol{\omega}) = \mathbf{0} \end{aligned} \tag{2.124}$$

Introduction of 2.123 and 2.124 into 2.121 yields

$$dm_i\mathbf{r}_i \times \ddot{\mathbf{r}}_G + dm_i\mathbf{k}\varepsilon r_i^2 = \mathbf{r}_i \times \mathbf{F}_i + \mathbf{r}_i \times \sum_{j=1}^{J} \mathbf{F}_{ij} \tag{2.125}$$

If one sum up such equations for all particles the body is made of, we can have

$$\sum_{i=1}^{I} dm_i\mathbf{r}_i \times \ddot{\mathbf{r}}_G + \sum_{i=1}^{I} dm_i\mathbf{k}\varepsilon r_i^2 = \sum_{i=1}^{I} \mathbf{r}_i \times \mathbf{F}_i + \sum_{i=1}^{I} (\mathbf{r}_i \times \sum_{j=1}^{J} \mathbf{F}_{ij}) \tag{2.126}$$

But

$$\sum_{i=1}^{I} dm_i\mathbf{k}\varepsilon r_i^2 = \mathbf{k}\varepsilon \sum_{i=1}^{I} r_i^2 dm_i = \mathbf{k}I_{zG}\varepsilon \tag{2.127}$$

Taking into account that $\frac{1}{m}\sum_{i}^{I=1} dm_i\mathbf{r}_i = \mathbf{0}$

$$\sum_{i=1}^{I} dm_i\mathbf{r}_i \times \ddot{\mathbf{r}}_G = (\sum_{i=1}^{I} dm_i\mathbf{r}_i) \times \ddot{\mathbf{r}}_G = 0 \tag{2.128}$$

Since $\mathbf{r}$ represents position vector of the particle i and therefore is independent from index of summation j,

$$\sum_{i=1}^{I}(\mathbf{r}_i\times\sum_{j=1}^{J}\mathbf{F}_{ij}) = \sum_{i=1}^{I}\sum_{j=1}^{J}(\mathbf{r}_i\times\mathbf{F}_{ij}) \tag{2.129}$$

The double sum contains only elements $\mathbf{r}_i\times\mathbf{F}_{ij} + \mathbf{r}_j\times\mathbf{F}_{ji}$. But

$$\mathbf{r}_i\times\mathbf{F}_{ij} + \mathbf{r}_j\times\mathbf{F}_{ji} = \mathbf{r}_i\times\mathbf{F}_{ij} - \mathbf{r}_j\times\mathbf{F}_{ij} = (\mathbf{r}_i - \mathbf{r}_j)\times\mathbf{F}_{ij} = \mathbf{r}_{ji}\times\mathbf{F}_{ij} = 0 \tag{2.130}$$

The above expression must be equal to zero since the two vectors $\mathbf{r}_{ji}$ and $\mathbf{F}_{ij}$are parallel (see Fig. 36) Introduction of the relationships from 2.127 to 2.130 into 2.126

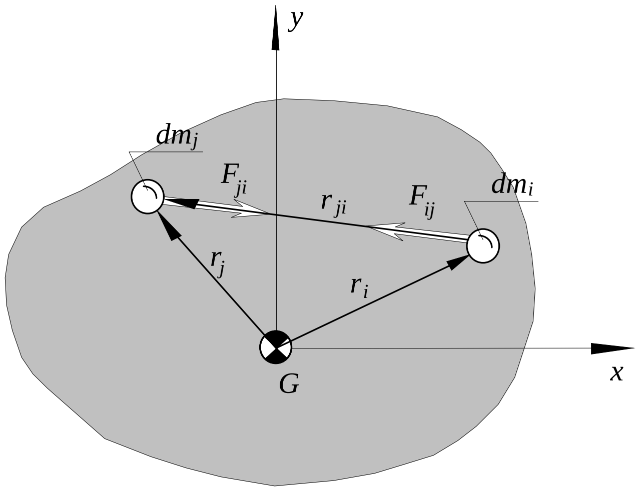


Figure 36

yields

$$\mathbf{k}I_{zG}\varepsilon = \sum_{i=1}^{I}\mathbf{r}_i\times\mathbf{F}_i \tag{2.131}$$

Multiplying the above expression by the unit vector $\mathbf{k}$ one can get

$$I_{zG}\varepsilon = \sum_{i=1}^{I}\mathbf{k}\cdot(\mathbf{r}_i\times\mathbf{F}_i) = M_{zG} \tag{2.132}$$

The equation 2.132 together with equation 2.119 forms set of equation that is known as the generalized Newton's equations

$$\boxed{\begin{array}{c} m\ddot{\mathbf{r}}_G = \mathbf{F} \\ I_{zG}\varepsilon = M_{zG} \end{array}} \tag{2.133}$$

The first equation is equivalent to two scalar equations which can be produced by multiplying it by two unit vectors associated with arbitrarily chosen system of coordinates xy. If one assume that the velocity of the centre of gravity is resolved along the body system of coordinates,

$$\mathbf{v}_G = \dot{\mathbf{r}}_G = \mathbf{i}(v_{Gx}) + \mathbf{j}(v_{Gy}) \tag{2.134}$$

the acceleration $\mathbf{a} = \ddot{\mathbf{r}}_G$ is

$$\begin{aligned}\ddot{\mathbf{r}}_G &= \mathbf{v}'_G + \boldsymbol{\omega} \times \mathbf{v_G} = \mathbf{i}(\dot{v}_{Gx}) + \mathbf{j}(\dot{v}_{Gy}) + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ v_{Gx} & v_{Gy} & 0 \end{vmatrix} = \\ &= \mathbf{i}(\dot{v}_{Gx} - v_{Gy}\omega) + \mathbf{j}(\dot{v}_{Gy} + v_{Gx}\omega) \end{aligned} \tag{2.135}$$

The mathematical model in this case takes the following form

$$\boxed{\begin{array}{c} m(\dot{v}_{Gx} - v_{Gy}\omega) = F_x \\ m(\dot{v}_{Gy} + v_{Gx}\omega) = F_y \\ I_{zG}\varepsilon = M_{zG} \end{array}} \tag{2.136}$$

Rotation of rigid body

Let us assume that a rigid body rotates about axis that is fixed in the inertial space. Without any damage to the generality of consideration, we may assume that the axis of rotation coincides with the axis Z of the inertial system of coordinates XYZ. Motion of the body is uniquely determined by the vector of the angular velocity $\boldsymbol{\omega}$ (see Fig. 37). Let us, as it was done in the previous section, split the body into individual particles. Each of them is governed by the following Newton's law.

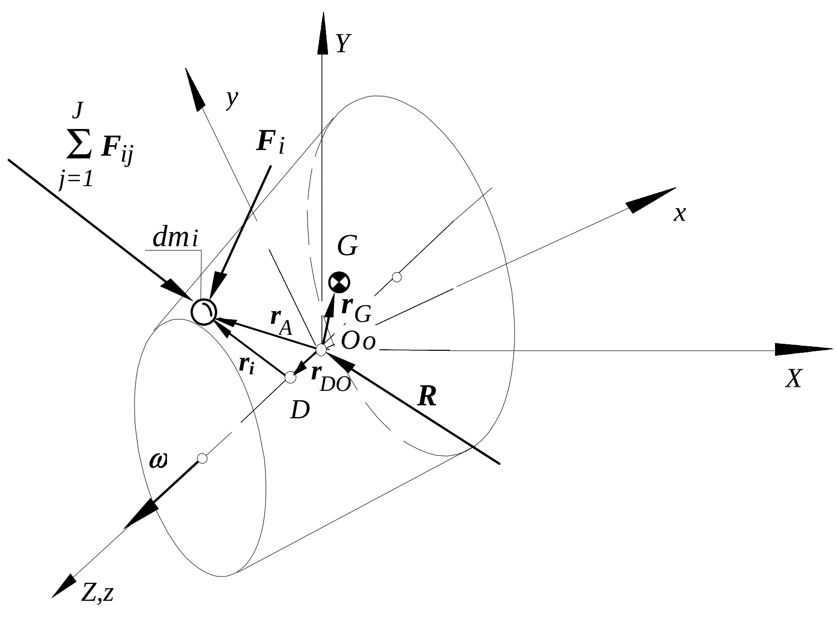


Figure 37

$$dm_i\ddot{\mathbf{r}}_A = \mathbf{F}_i + \sum_{j=1}^{J} \mathbf{F}_{ij} \tag{2.137}$$

In the above equation $\mathbf{r}_A$ stands for the absolute position vector. According to the Fig. 37 it can be expressed as follows

$$\mathbf{r}_A = \mathbf{r}_{DO} + \mathbf{r}_i \tag{2.138}$$

Two subsequent differentiations with respect to time produce the wanted absolute acceleration of the particle i.

$$\dot{\mathbf{r}}_A = \boldsymbol{\omega} \times \mathbf{r}_i \tag{2.139}$$

$$\ddot{\mathbf{r}}_A = \dot{\boldsymbol{\omega}} \times \mathbf{r}_i + \boldsymbol{\omega} \times \dot{\mathbf{r}}_i \tag{2.140}$$

Introduction of the above expression into equation 2.137 yields

$$dm_i \dot{\boldsymbol{\omega}} \times \mathbf{r}_i + dm_i \boldsymbol{\omega} \times \dot{\mathbf{r}}_i = \mathbf{F}_i + \sum_{j=1}^{J} \mathbf{F}_{ij} \tag{2.141}$$

By multiplying the left and right hand side of equation 2.141 by the vector $\mathbf{r}_i$ we are getting.

$$\mathbf{r}_i \times (dm_i \dot{\boldsymbol{\omega}} \times \mathbf{r}_i + dm_i \boldsymbol{\omega} \times \dot{\mathbf{r}}_i) = \mathbf{r}_i \times (\mathbf{F}_i + \sum_{j=1}^{J} \mathbf{F}_{ij}) \tag{2.142}$$

or

$$dm_i \mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) + dm_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) = \mathbf{r}_i \times \mathbf{F}_i + \mathbf{r}_i \times \sum_{j=1}^{J} \mathbf{F}_{ij} \tag{2.143}$$

In the same manner as it was done in previous section we can show that

$$\mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i) = \mathbf{k} \varepsilon r_i^2 \tag{2.144}$$

and

$$\mathbf{r}_i \times (\boldsymbol{\omega} \times \dot{\mathbf{r}}_i) = \mathbf{0} \tag{2.145}$$

Hence,

$$dm_i \mathbf{k} \varepsilon r_i^2 = \mathbf{r}_i \times \mathbf{F}_i + \mathbf{r}_i \times \sum_{j=1}^{J} \mathbf{F}_{ij} \tag{2.146}$$

If one sum up such equations for all particles the body is made of, we can have

$$\sum_{i=1}^{I} dm_i \mathbf{k} \varepsilon r_i^2 = \sum_{i=1}^{I} \mathbf{r}_i \times \mathbf{F}_i + \sum_{i=1}^{I} (\mathbf{r}_i \times \sum_{j=1}^{J} \mathbf{F}_{ij}) \tag{2.147}$$

But

$$\sum_{i=1}^{I} dm_i \mathbf{k} \varepsilon r_i^2 = \mathbf{k} \varepsilon \sum_{i=1}^{I} r_i^2 dm_i = \mathbf{k} I_{zO} \varepsilon \tag{2.148}$$

Consideration similar to the one carried out in the previous section leads to conclusion that

$$\sum_{i=1}^{I} (\mathbf{r}_i \times \sum_{j=1}^{J} \mathbf{F}_{ij}) = \sum_{i=1}^{I} \sum_{j=1}^{J} (\mathbf{r}_i \times \mathbf{F}_{ij}) = 0 \tag{2.149}$$

Introduction of 2.148 and 2.149 into 2.150 yields

$$\mathbf{k}I_{zO}\varepsilon = \sum_{i=1}^{I} \mathbf{r}_i \times \mathbf{F}_i \tag{2.150}$$

Multiplying the above equation by the unit vector $\mathbf{k}$ one can get

$$I_{zO}\varepsilon = \sum_{i=1}^{I} \mathbf{k} \cdot (\mathbf{r}_i \times \mathbf{F}_i) = M_{zO} \tag{2.151}$$

where I_z is the mass moment of inertia of the body about the axis of rotation z and M_z stands for the component of the resultant moment due to all the external forces acting on the body along axis of rotation z.

Hence

$$\boxed{I_{zO}\varepsilon = M_{zO}} \tag{2.152}$$

The above equation governs the rotation of the rigid body and allows the absolute angular acceleration ε to be determined as a function of time. Now, the absolute linear acceleration of the centre of gravity G can be obtained. It is the second derivative of the absolute position vector $\mathbf{r}_G$

$$\dot{\mathbf{r}}_G = \mathbf{r}'_G + \boldsymbol{\omega} \times \mathbf{r}_G = \boldsymbol{\omega} \times \mathbf{r}_G \tag{2.153}$$

$$\ddot{\mathbf{r}}_G = \dot{\boldsymbol{\omega}} \times \mathbf{r}_G + \boldsymbol{\omega} \times \dot{\mathbf{r}}_G = \dot{\boldsymbol{\omega}} \times \mathbf{r}_G + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{\mathbf{G}}) = \boldsymbol{\varepsilon} \times \mathbf{r}_G + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{\mathbf{G}}) \tag{2.154}$$

where

$$\begin{aligned} \boldsymbol{\varepsilon} &= \mathbf{k}\varepsilon \\ \boldsymbol{\omega} &= \mathbf{k}\int_0^t \varepsilon(\tau)d\tau \end{aligned} \tag{2.155}$$

In the case considered, the resultant of all the external forces contains the unknown interaction force $\mathbf{R}$ between the rotating body and the inertial space. This force does not appear in equation 2.152 since its line of action intersects the axis of rotation. Components of this force can be now obtained from equation

$$m\ddot{\mathbf{r}}_G = \mathbf{F} = \mathbf{R} + \mathbf{P} \tag{2.156}$$

2.2.4 *Problems*

Moments of inertia

Problem 24

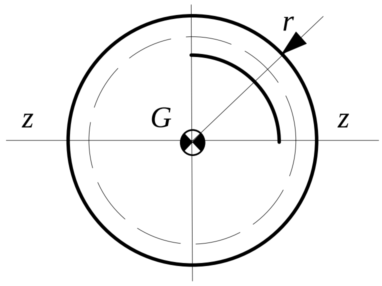


Figure 38

Derive the expression for the moment of inertia of the spherical shell of the mass m and radius r about the axis z through its centre of gravity G (see Fig. 38).

Solution

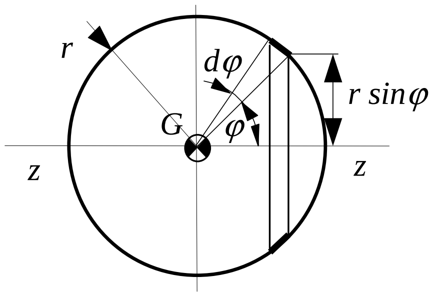


Figure 39

A shell is the solid which thickness is negligible with respect to its other dimensions (in this case the radius r). The unit mass of the shall considered (mass per unit of surface) is

$$\rho = \frac{m}{4\pi r^2} \tag{2.157}$$

Let us consider the ring shown in Fig. 39. Its area

$$dA = (2\pi r \sin\varphi)(rd\varphi) \tag{2.158}$$

Hence its mass is

$$dm = \rho dA = \frac{m}{4\pi r^2} 2\pi r^2 \sin\varphi d\varphi = \frac{1}{2} m \sin\varphi d\varphi \tag{2.159}$$

Its moment of inertia about the axis z is

$$dI_z = (r\sin\varphi)^2 dm = \frac{1}{2} mr^2 \sin^3\varphi d\varphi \tag{2.160}$$

The integration of the above expression with respect to φ within the limits from $\varphi = 0$ to $\varphi = \pi$ yields the wanted moment of inertia of the shell.

$$I_z = \int_0^\pi \frac{1}{2} mr^2 \sin^3\varphi d\varphi = \frac{1}{2} mr^2 \int_0^\pi \sin^3\varphi d\varphi = \frac{1}{2} mr^2 \frac{4}{3} = \frac{2}{3} mr^2 \tag{2.161}$$

Problem 25

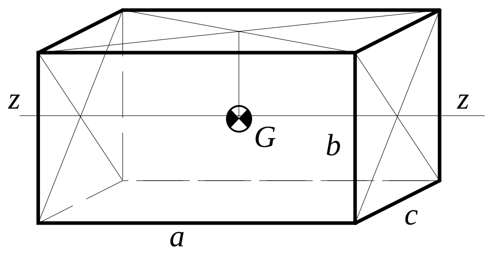


Figure 40

Derive the expression for the moment of inertia of the rectangular block of the mass m (see Fig.40) about the axis z through its centre of gravity G.

Answer:

$I_G = \frac{1}{12} m(b^2 + c^2)$

Problem 26

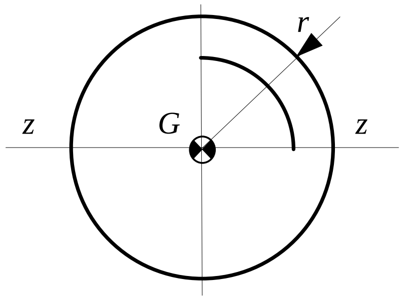


Figure 41

Derive the expression for the moment of inertia of the spherical ball of the mass m and radius r about the axis z through its centre of gravity G (see Fig. 41)

Answer:

$I_G = \frac{2}{5}mr^2$

Problem 27

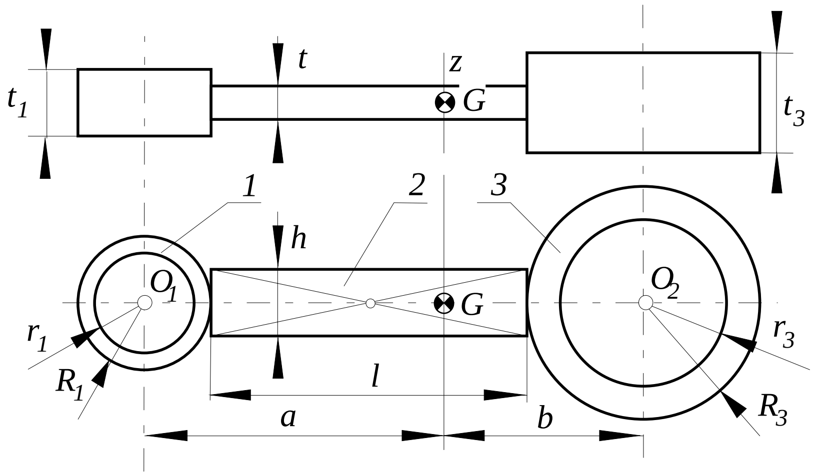


Figure 42

A connecting rod was approximated by the rectangular block and the two rings shown in Fig. 42. Density of the connecting rod is d. Produce the expression for the moment of inertia about the axis $z - z$ through the centre of gravity of the connecting rod.

Solution

The mass of the ring 1.

$$m_1 = m_{R1} - m_{r1} = \pi t_1 d(R_1^2 - r_1^2) \tag{2.162}$$

The moment of inertia of the ring 1 about axis through O_1.

$$I_1 = I_{R1} - I_{r1} = \frac{m_{R1}R_1^2}{2} - \frac{m_{r1}r_1^2}{2} = \frac{1}{2}\left(\pi R_1^4 t_1 d - \pi r_1^4 t_1 d\right) = \frac{\pi t_1 d}{2}\left(R_1^4 - r_1^4\right) \tag{2.163}$$

The mass of the rectangular block 2

$$m_2 = lhtd \tag{2.164}$$

The moment of inertia of the rectangular block about axis through the point O.

$$I_2 = \frac{m_2}{12}\left(l^2 + h^2\right) = \frac{lhtd}{12}\left(l^2 + h^2\right) \tag{2.165}$$

The mass of the ring 3.

$$m_3 = m_{R3} - m_{r3} = \pi t_3 d(R_3^2 - r_3^2) \tag{2.166}$$

The moment of inertia of the ring 3 about axis through O_3.

$$I_3 = I_{R3} - I_{r3} = \frac{m_{R3}R_3^2}{2} - \frac{m_{r3}r_3^2}{2} = \frac{1}{2}\left(\pi R_3^4 t_3 d - \pi r_3^4 t_3 d\right) = \frac{\pi t_3 d}{2}\left(R_3^4 - r_3^4\right) \tag{2.167}$$

The total mass of the connecting rod.

$$m = m_1 + m_2 + m_3 = \pi t_1 d(R_1^2 - r_1^2) + lhtd + \pi t_3 d(R_3^2 - r_3^2) \tag{2.168}$$

The position of the centre of gravity G.

$$ma = m_1 0 + m_2\left(R_1 + \frac{l}{2}\right) + m_3\left(R_1 + l + R_3\right) \tag{2.169}$$

Hence

$$a = \frac{m_2\left(R_1 + \frac{l}{2}\right) + m_3\left(R_1 + l + R_3\right)}{m} \tag{2.170}$$

The moment of inertia of the ring 1 about axis through the point G.

$$I_{1G} = I_1 + m_1 a^2 \tag{2.171}$$

The moment of inertia of the block 2 about axis through the point G.

$$I_{2G} = I_2 + m_2\left(a - R_1 - \frac{l}{2}\right)^2 \tag{2.172}$$

The moment of inertia of the ring 3 about axis through the point G.

$$I_{3G} = I_3 + m_3 b^2 = I_3 + m_3\left(R_1 + l + R_3 - a\right)^2 \tag{2.173}$$

The total moment of inertia about axis through the point G.

$$I_G = I_{1G} + I_{2G} + I_{3G} \tag{2.174}$$

Problem 28

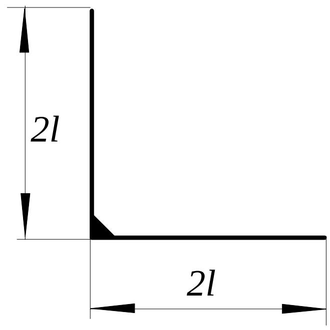


Figure 43

Two uniform rods, each of mass m and length $2l$, are joined together to form the element shown in Fig. 43. Produce the expression for the moment of inertia of the element about the axis through its centre of gravity and perpendicular to both rods.

Answer:
$I_G = \frac{5}{3}ml^2$

Problem 29

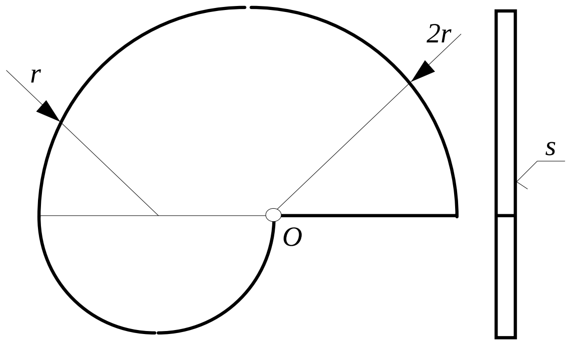


Figure 44

The flat and rigid element shown in Fig. 44 has mass m. Develop the formula for the moment of inertia of this element about the axis through the point O and perpendicular to its surface s.

Answer:
$I_G = 1.9mr^2$

Problem 30

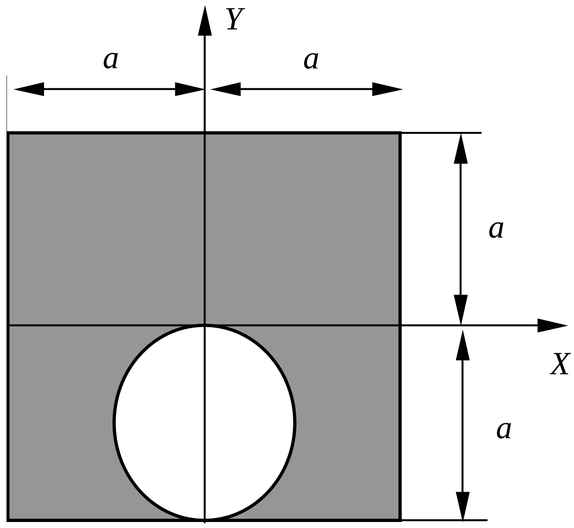


Figure 45

Fig. 45 shows a flat, uniform and rigid body of mass m. Produce the expression for the moment of inertia of the body about the axis through its centre of gravity and perpendicular to the plane XY.

Answer:

$$I_{Z_G} = \left(\frac{16^2 - 9\pi}{24(16-\pi)} - \left(\frac{\frac{1}{2}\pi}{16-\pi} \right)^2 \right) ma^2 = 0.7229ma^2$$

Kinetic energy

Problem 31

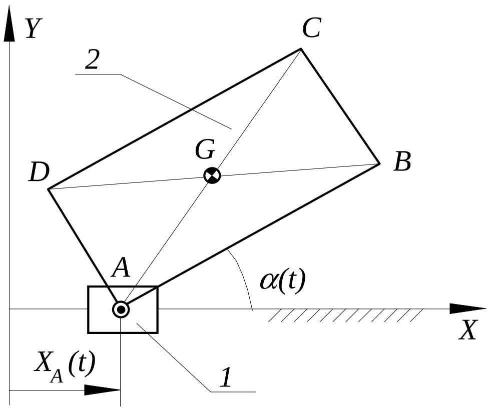


Figure 46

The link 1 of the system shown in Fig 46 moves along the axis X of the inertial system of coordinates XY and its motion is given by the following function of time $X_A(t)$. The rectangular block 2 of mass m is hinged to the link1 at the point A. The relative motion of the block 2 with respect to the link 1 is given as a function of time $\alpha(t)$. Produce the expression for the kinetic energy of the rectangular block 2.

Solution

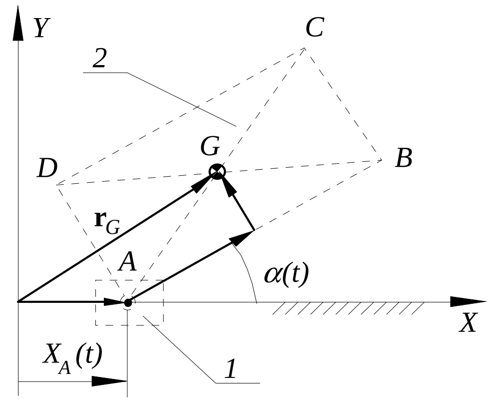


Figure 47

The block 2 performs a general motion. Its kinetic energy is.

$$T = \frac{1}{2}v_G^2 m + \frac{1}{2}I_{zG}\omega^2 \tag{2.175}$$

The absolute velocity of the centre of gravity of the block is the first time derivative of the following absolute position vector $\mathbf{r}_G$ (see Fig. 47).

$$\mathbf{r}_G = \mathbf{I}(X_A + \frac{a}{2}\cos\alpha - \frac{b}{2}\sin\alpha) + \mathbf{J}(\frac{a}{2}\sin\alpha + \frac{b}{2}\cos\alpha) \tag{2.176}$$

Hence, its absolute velocity is

$$\mathbf{v}_G = \mathbf{I}(\dot{X}_A - \frac{a}{2}\dot{\alpha}\sin\alpha - \frac{b}{2}\dot{\alpha}\cos\alpha) + \mathbf{J}(\frac{a}{2}\dot{\alpha}\cos\alpha - \frac{b}{2}\dot{\alpha}\sin\alpha) \tag{2.177}$$

It follows that

$$v_G^2 = v_{GX}^2 + v_{GY}^2 = (\dot{X}_A - \frac{a}{2}\dot{\alpha}\sin\alpha - \frac{b}{2}\dot{\alpha}\cos\alpha)^2 + (\frac{a}{2}\dot{\alpha}\cos\alpha - \frac{b}{2}\dot{\alpha}\sin\alpha)^2 \tag{2.178}$$

The absolute angular velocity of the block is

$$\omega = \dot{\alpha} \tag{2.179}$$

The moment of inertia of the block about the axis through its gravity centre is

$$I_{zG} = \frac{1}{12}m(a^2 + b^2) \tag{2.180}$$

Eventually, the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}m\left((\dot{X}_A - \frac{a}{2}\dot{\alpha}\sin\alpha - \frac{b}{2}\dot{\alpha}\cos\alpha)^2 + (\frac{a}{2}\dot{\alpha}\cos\alpha - \frac{b}{2}\dot{\alpha}\sin\alpha)^2\right) + \frac{1}{24}m(a^2 + b^2)\dot{\alpha}^2 = \\ &= \frac{1}{2}m\dot{X}_A^2 - \frac{1}{2}m\dot{X}_A a\dot{\alpha}\sin\alpha - \frac{1}{2}m\dot{X}_A b\dot{\alpha}\cos\alpha + \frac{1}{6}m\dot{\alpha}^2 a^2 + \frac{1}{6}m\dot{\alpha}^2 b^2 \end{aligned} \tag{2.181}$$

Problem 32

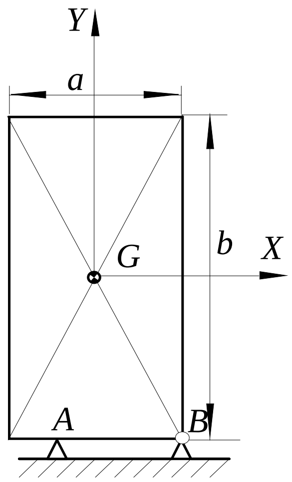


Figure 48

The rectangular block of mass m shown in Fig. 48 is supported at the point A and hinged at the point B to the motionless floor. Produce the expression for the initial angular velocity of the block that is necessary to overturn it.

Solution

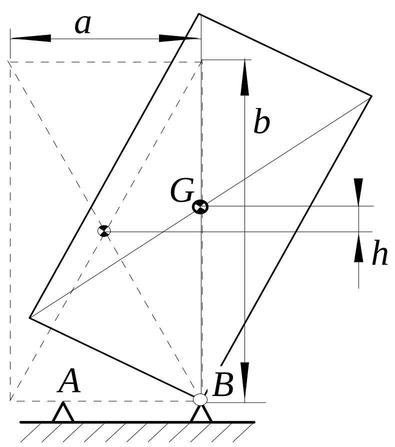


Figure 49

To overturn the block, the initial kinetic energy must be equal or greater than the increment in the potential energy between the two positions shown in Fig. 49. The increment is.

$$\Delta V = mgh = \frac{1}{2}mg\left(\sqrt{a^2+b^2}-b\right) \tag{2.182}$$

Since the block perform the rotational motion about the centre of rotation B, its kinetic energy is

$$T = \frac{1}{2}I_B\omega^2 \tag{2.183}$$

The moment of inertia of the block about the axis through its centre of gravity is

$$I_G = \frac{1}{12}m\left(a^2+b^2\right) \tag{2.184}$$

Application of the parallel axes theorem yields the moment of inertia about the axis through the point of rotation B

$$I_B = I_G + m\left(\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2\right) = \frac{1}{3}m\left(a^2+b^2\right) \tag{2.185}$$

It follows that

$$T = \frac{1}{2}\frac{1}{3}m\left(a^2+b^2\right)\omega^2 = \Delta V = \frac{1}{2}mg\left(\sqrt{a^2+b^2}-b\right) \tag{2.186}$$

Hence, the initial velocity of the block should be greater than

$$\omega = \sqrt{\frac{g\left(\sqrt{a^2+b^2}-b\right)}{\frac{1}{3}\left(a^2+b^2\right)}} \tag{2.187}$$

Problem 33

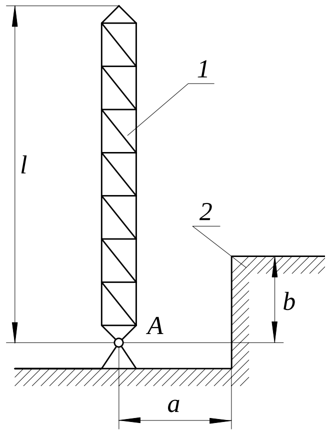


Figure 50

The mast 1 of the length l and mass m (see Fig.50) can be considered as a uniform rod. The mast is leaving its equilibrium position with the initial angular velocity equal to zero to strike the wall 2. Produce the expression for the angular velocity of the mast when it strikes the wall.

Answer:

$$\omega = \sqrt{\frac{3g\left(1-\frac{b}{\sqrt{a^2+b^2}}\right)}{l}}$$

Problem 34

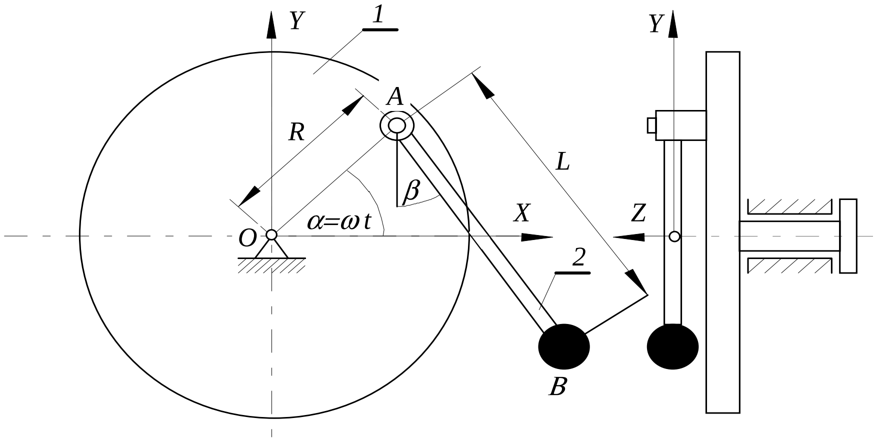


Figure 51

The link 1 (see Fig. 51) rotates with the constant angular velocity ω about the horizontal axis Z of the inertial system of coordinates XYZ. The link 2 is hinged at the point A to the link 1. Its angular motion is determined by the angular displacement $\beta(t)$. The link 2 is assembled of the uniform rod AB of mass M and the particle B of mass m. Produce the expression for the kinetic energy of the link 2.

Answer:

$$T = \frac{ML^2}{24}\dot{\beta}^2 + \frac{1}{2}M\left[\left(-R\omega\sin\omega t + \frac{L}{2}\dot{\beta}\cos\beta\right)^2 + \left(R\omega\cos\omega t + \frac{L}{2}\dot{\beta}\sin\beta\right)^2\right] + \frac{1}{2}m\left[\left(-R\omega\sin\omega t + L\dot{\beta}\cos\beta\right)^2 + \left(R\omega\cos\omega t + L\dot{\beta}\sin\beta\right)^2\right]$$

Problem 35

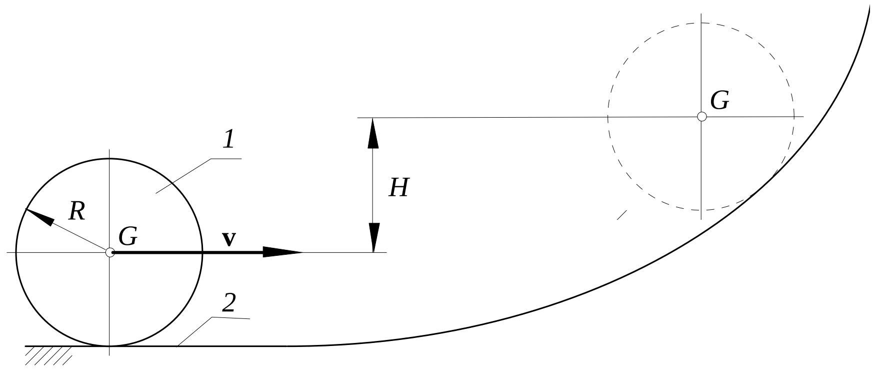


Figure 52

The cylinder of mass m and radius R roles over the slide 2 without slipping (see Fig. 52). At the position 1 the linear velocity of its centre of gravity G is $\mathbf{v}$. The cylinder comes to a rest at the position shown with the dashed line. Derive the expression for the distance H.

Answer:

$v = \sqrt{\frac{4}{3}gH}$

Problem 36

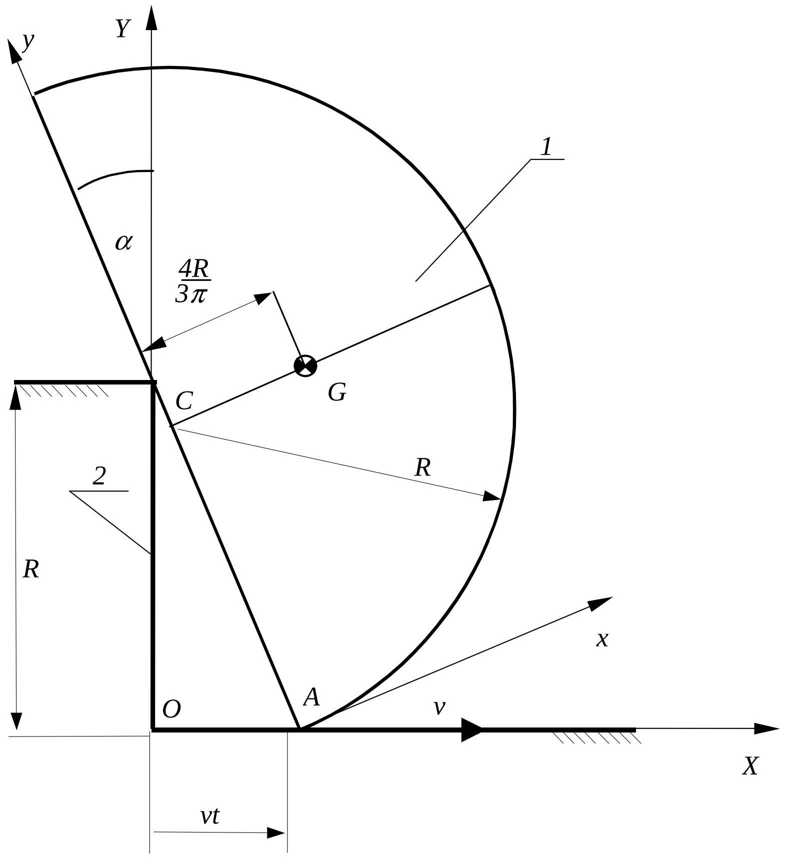


Figure 53

The semi-cylinder 1 of mass m and radius R (see Fig. 53) is supported by the wall 2. Its edge A moves along the horizontal axis X with a constant velocity v.

Produce:

1. the expression for the moment of inertia of the semi-cylinder 1 about the axis through its centre of gravity G.

Answer:

$I_G = \frac{1}{2}mR^2 + m\left(\frac{4R}{3\pi}\right)^2$

2. the expression for the absolute angular velocity of the semi-cylinder 1

Answer:

$\omega = \frac{vR}{R^2+v^2t^2}$

3. the expression for the components of the absolute velocity of the centre of gravity G of the semi-cylinder 1 along the inertial system of coordinates XYZ

Answer:

$\mathbf{v}_G = \mathbf{I}\left(v - R\dot{\alpha}\cos\alpha - \frac{4R}{3\pi}\dot{\alpha}\sin\alpha\right) + \mathbf{J}\left(-R\dot{\alpha}\sin\alpha + \frac{4R}{3\pi}\dot{\alpha}\cos\alpha\right)$

4. the expression for the kinetic energy of the semi-cylinder 1.

Answer:

$T = \frac{1}{2}I_G\omega^2 + \frac{1}{2}m\mathbf{v}_G^2$

Equations of motion

Problem 37

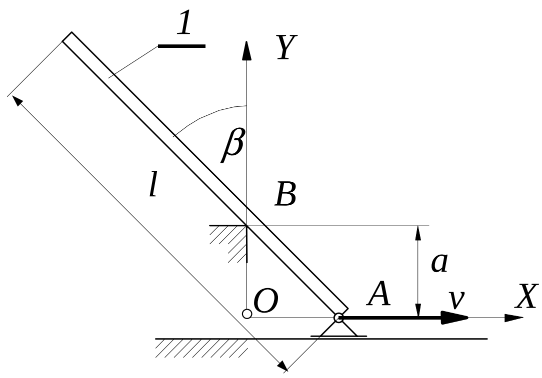


Figure 54

The rigid and uniform rod 1 of mass m moves in the vertical plane of the inertial system of coordinates XYZ (see Fig. 54). Its end A moves along the horizontal axis X with the constant velocity v. Produce expression for the reactions at the constraints A and B.

Solution

According to the constraints shown in Fig. 55, the following vectorial equation has to be fulfilled

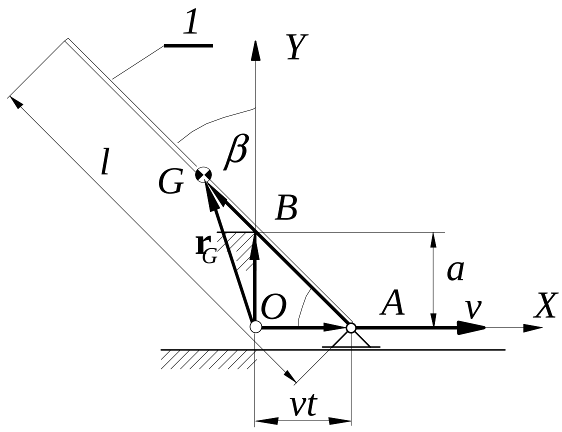


Figure 55

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \tag{2.188}$$

Hence

$$\mathbf{I}vt + \mathbf{I}(-AB\sin\beta) + \mathbf{J}(AB\cos\beta) = \mathbf{J}a \tag{2.189}$$

This equation is equivalent to two scalar equations

$$AB\sin\beta = vt \tag{2.190}$$
$$AB\cos\beta = a \tag{2.191}$$

Solving them for the unknown AB and β one can get

$$\beta = \arctan\frac{vt}{a} \tag{2.192}$$
$$AB = \sqrt{v^2t^2 + a^2} \tag{2.193}$$

The following subsequent differentiation of the first of the above equation yields the angular velocity and the angular acceleration respectively

$$\omega = \dot{\beta} = \frac{\frac{v}{a}}{1 + \left(\frac{v}{a}\right)^2 t^2} \tag{2.194}$$

$$\varepsilon = \dot{\omega} = \frac{-2\left(\frac{v}{a}\right)^3 t}{\left(1 + \left(\frac{v}{a}\right)^2 t^2\right)^2} \tag{2.195}$$

The absolute acceleration of the centre of gravity G can be obtained by differentiation of the position vector of the gravity centre G.

$$\mathbf{r}_G = \mathbf{I}\left(vt - \left(\frac{l}{2}\sin\beta\right)\right) + \mathbf{J}\left(\frac{l}{2}\cos\beta\right) \tag{2.196}$$

Hence

$$\mathbf{v}_G = \mathbf{I}\left(v - \left(\frac{l}{2}\dot{\beta}\cos\beta\right)\right) + \mathbf{J}\left(-\frac{l}{2}\dot{\beta}\sin\beta\right) \tag{2.197}$$

$$\mathbf{a}_G = \mathbf{I}\left(-\frac{l}{2}\ddot{\beta}\cos\beta + \frac{l}{2}\dot{\beta}^2\sin\beta\right) + \mathbf{J}\left(-\frac{l}{2}\ddot{\beta}\sin\beta - \frac{l}{2}\dot{\beta}^2\cos\beta\right) \tag{2.198}$$

Let us apply the generalized Newton's law

$$\begin{aligned} m\mathbf{a}_G &= \mathbf{F} \\ I_G\varepsilon &= M_G \end{aligned} \tag{2.199}$$

to the free body diagram shown in Fig. 56.

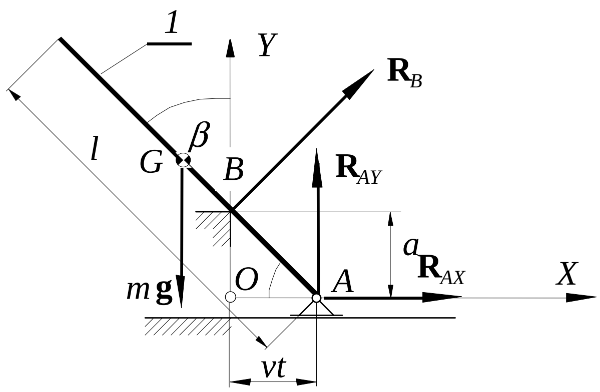


Figure 56

In the above equations I_G stands for the moment of inertia about axis through the centre of gravity G.

$$I_G = \frac{ml^2}{12} \tag{2.200}$$

Introduction of the previously developed expression into the equation 2.199 yields three algebraic linear equations with three unknown components of the interaction forces R_B, R_{AX}, R_{AY}.

$$\begin{aligned} \left(-\frac{l}{2}\ddot{\beta}\cos\beta + \frac{l}{2}\dot{\beta}^2\sin\beta\right)m &= R_{AX} + R_B\cos\beta \\ \left(-\frac{l}{2}\ddot{\beta}\sin\beta - \frac{l}{2}\dot{\beta}^2\cos\beta\right)m &= -mg + R_B\sin\beta + R_{AY} \\ \varepsilon\frac{ml^2}{12} &= R_{AY}\frac{l}{2}\sin\beta + R_{AX}\frac{l}{2}\cos\beta + R_B\left(\frac{l}{2} - \frac{a}{\cos\beta}\right) \end{aligned} \tag{2.201}$$

Problem 38

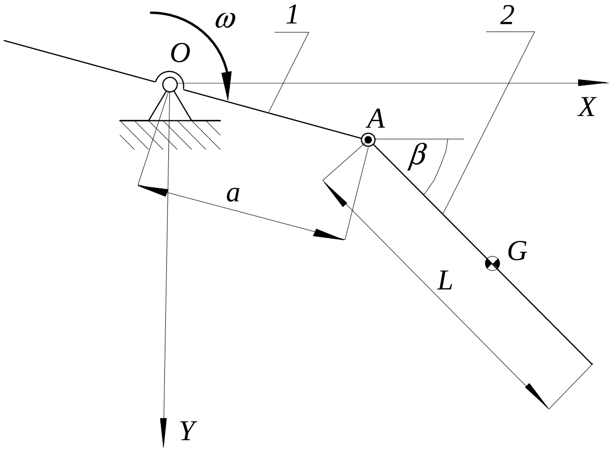


Figure 57

The link 1 of the mechanism shown in Fig. 57 rotates with the constant angular velocity ω. The link 2 is hinged to the link 1 at the point A. Its mass is m,moment of inertia about axis through the centre of gravity G is I_G and its length is L. Produce the differential equations of motion of the link 2.

Solution

Motion of the link 2 is governed by the generalized Newton's equations.

$$\begin{aligned} \mathbf{a}_G m &= \mathbf{F} \qquad (2.202)\\ \varepsilon I_G &= M_G \qquad (2.203) \end{aligned}$$

The acceleration of the centre of gravity G can be obtained by differentiation of the absolute position vector $\mathbf{r}_G$ (see Fig. 58).

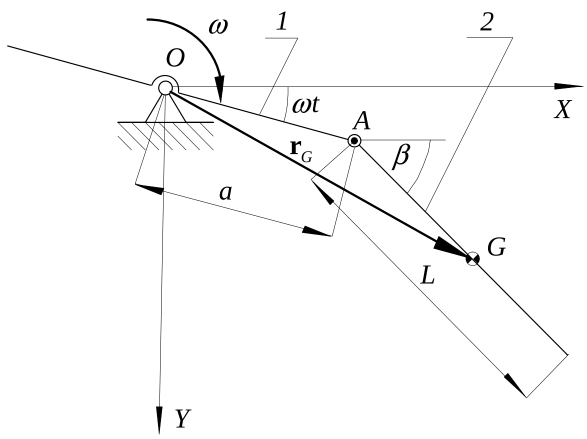


Figure 58

$$\mathbf{r}_G = \mathbf{I}(a\cos\omega t + \frac{L}{2}\cos\beta) + \mathbf{J}(a\sin\omega t + \frac{L}{2}\sin\beta) \qquad (2.204)$$

Hence, the absolute velocity of the centre of gravity is

$$\mathbf{v}_G = \dot{\mathbf{r}}_G = \mathbf{I}(-a\omega\sin\omega t - \frac{L}{2}\dot{\beta}\sin\beta) + \mathbf{J}(a\omega\cos\omega t + \frac{L}{2}\dot{\beta}\cos\beta) \qquad (2.205)$$

and its acceleration is

$$\begin{aligned} \mathbf{a}_G &= \ddot{\mathbf{r}}_G = \\ &= \mathbf{I}(-a\omega^2\cos\omega t - \frac{L}{2}\ddot{\beta}\sin\beta - \frac{L}{2}\dot{\beta}^2\cos\beta) + \mathbf{J}(-a\omega^2\sin\omega t + \frac{L}{2}\ddot{\beta}\cos\beta - \frac{L}{2}\dot{\beta}^2\sin\beta) \end{aligned} \qquad (2.206)$$

To produce the right hand sides of the equation 2.202 it is necessary to create the free body diagram for the link 2. It is shown in Fig. 59. Now, the equations of motion can be rewritten as follows

$$\begin{aligned} \left(-a\omega^2\cos\omega t - \frac{L}{2}\ddot{\beta}\sin\beta - \frac{L}{2}\dot{\beta}^2\cos\beta\right)m &= R_X \\ \left(-a\omega^2\sin\omega t + \frac{L}{2}\ddot{\beta}\cos\beta - \frac{L}{2}\dot{\beta}^2\sin\beta\right)m &= R_Y + mg \qquad (2.207)\\ \ddot{\beta}I_G &= R_X\frac{L}{2}\sin\beta - R_Y\frac{L}{2}\cos\beta \end{aligned}$$

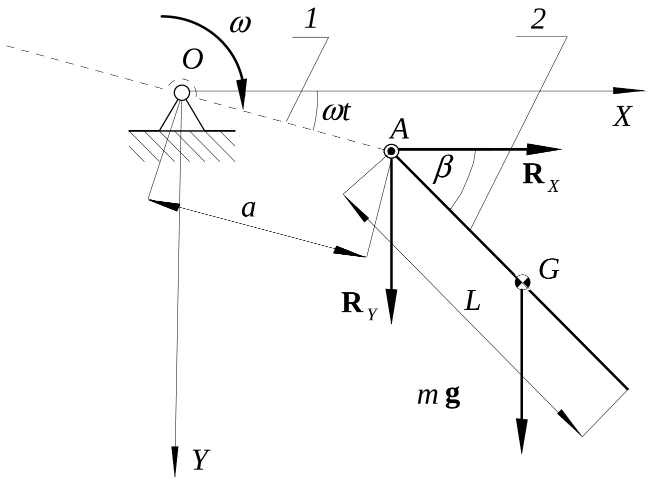


Figure 59

The first two equations of the above set, allow the components of the reaction $\mathbf{R}$ to be expressed in terms of the angular displacement β.

$$\begin{aligned} R_X &= \left(-a\omega^2\cos\omega t - \frac{L}{2}\ddot{\beta}\sin\beta - \frac{L}{2}\dot{\beta}^2\cos\beta\right)m \\ R_Y &= \left(-a\omega^2\sin\omega t + \frac{L}{2}\ddot{\beta}\cos\beta - \frac{L}{2}\dot{\beta}^2\sin\beta\right)m - mg \end{aligned} \tag{2.208}$$

If we introduce the expressions 2.208 into the last equation of the set 2.207 we are getting the following differential equation with the unknown β.

$$\begin{aligned} \ddot{\beta}I_G &= \left(-a\omega^2\cos\omega t - \frac{L}{2}\ddot{\beta}\sin\beta - \frac{L}{2}\dot{\beta}^2\cos\beta\right)m\frac{L}{2}\sin\beta \\ &\quad - \left(\left(-a\omega^2\sin\omega t + \frac{L}{2}\ddot{\beta}\cos\beta - \frac{L}{2}\dot{\beta}^2\sin\beta\right)m - mg\right)\frac{L}{2}\cos\beta \end{aligned} \tag{2.209}$$

After a simplification, the above equation takes the following form.

$$\ddot{\beta}\left(I_G + m\frac{L^2}{4}\right) - \frac{L}{2}m\left(a\omega^2\sin\left(\omega t - \beta\right) + g\cos\beta\right) = 0 \tag{2.210}$$

This is a non-linear differential equation and can be solved by means of the numerical methods only. Inserting this solution into the functions 2.208 one can obtain the reactions as the explicit functions of time.

Problem 39

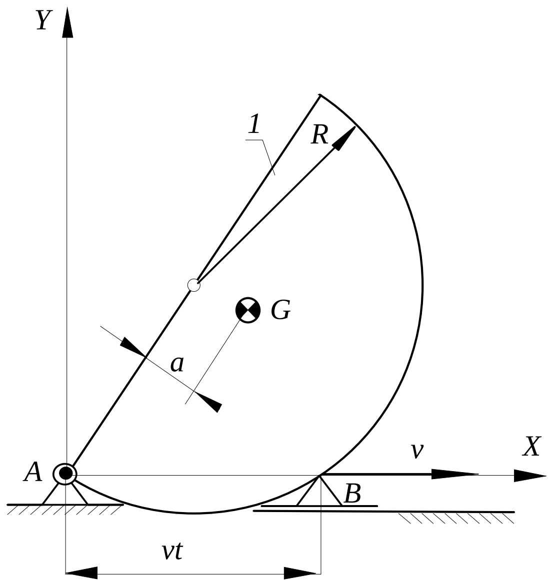


Figure 60

The rigid body *1* shown in Fig 60 is free to rotate about the horizontal axis Z of the inertial system of coordinates XYZ. The support A is stationary whereas the support B moves along the axis X with the constant velocity v. The mass of the rigid body is m and its moment of inertia about the axis through its centre of gravity G and parallel to Z is I_G.

Produce

1. the expression for the absolute angular velocity of the rigid body
2. the expression for the absolute velocity and the absolute acceleration of the centre of gravity of the rigid body
3. the expression for the kinetic energy of the rigid body.
4. the expression for the interaction forces at the supports A and B.
5. the expression for the force that must be applied to the support B to assure motion with the assume velocity v.

Solution

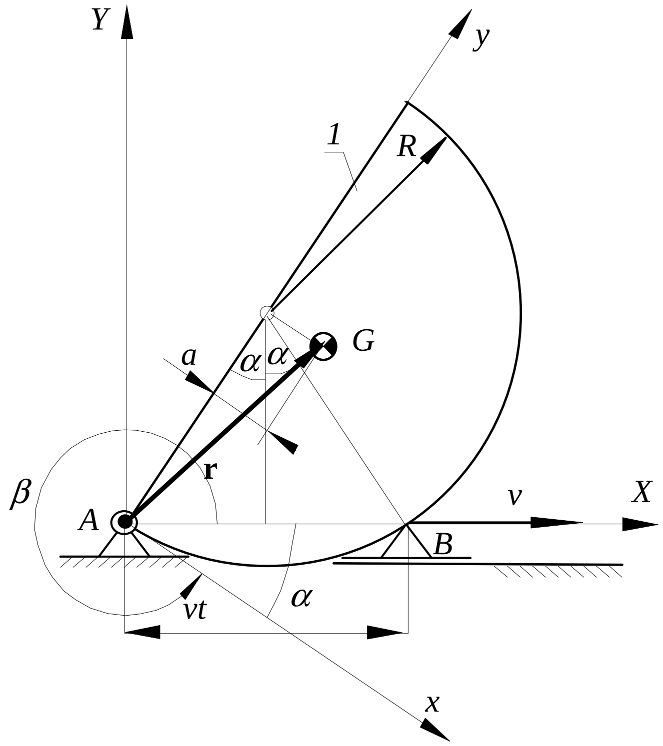


Figure 61

It is easy to see from the Fig. 61 that sinus of the angle between the absolute system of coordinates XYZ and the body system of coordinates xyz is

$$\sin\alpha = \frac{vt}{2R} \tag{2.211}$$

Hence, the angular displacement α is determined by the following function

$$\alpha = \arcsin\frac{vt}{2R} \tag{2.212}$$

By definition, the angular velocity of the body 1 is equal to the angular velocity of the body system of coordinates xyz. This, in turn, is equal to the first derivative of the angular displacement β between axis X and x measured in the direction determined by the right handed screw rule.

$$\omega = \frac{d}{dt}(\beta) = \frac{d}{dt}(2\pi - \alpha) = -\dot{\alpha} = -\frac{\frac{1}{2}\frac{v}{R}}{\sqrt{\left(1 - \frac{1}{4}v^2\frac{t^2}{R^2}\right)}} = -\frac{v}{\sqrt{(4R^2 - v^2t^2)}} \tag{2.213}$$

To produce the absolute velocity of the point G one has to develop the expression for the absolute position vector $\mathbf{r}$. According to the Fig.61 this position vector is

$$\mathbf{r} = \mathbf{I}(R\sin\alpha + a\cos\alpha) + \mathbf{J}(R\cos\alpha - a\sin\alpha) \tag{2.214}$$

Differentiation of the above vector yields the expression for the absolute velocity and the absolute acceleration

$$\mathbf{v} = \dot{\mathbf{r}} = \mathbf{I}(R\dot{\alpha}\cos\alpha - a\dot{\alpha}\sin\alpha) + \mathbf{J}(-R\dot{\alpha}\sin\alpha - a\dot{\alpha}\cos\alpha) \tag{2.215}$$

$$\begin{aligned} \mathbf{a} &= \dot{\mathbf{v}} = \mathbf{I}(R\ddot{\alpha}\cos\alpha - R\dot{\alpha}^2\sin\alpha - a\ddot{\alpha}\sin\alpha - a\dot{\alpha}^2\cos\alpha) \\ &\quad +\mathbf{J}(-R\ddot{\alpha}\sin\alpha - R\dot{\alpha}^2\cos\alpha - a\ddot{\alpha}\cos\alpha + a\dot{\alpha}^2\sin\alpha) \end{aligned} \tag{2.216}$$

The body 1 performs the rotation about the point A. Therefore its kinetic energy is

$$T = \frac{1}{2}I_A\omega^2 = \frac{1}{2}I_A\left(\frac{v}{\sqrt{(4R^2 - v^2t^2)}}\right)^2 \tag{2.217}$$

According to the parallel axes theorem, the moment of inertia of the body 1 about the point A is

$$I_A = I_G + m(AG)^2 = I_G + m(R^2 + a^2) \tag{2.218}$$

Hence

$$T = \frac{1}{2}(I_G + m(R^2 + a^2))\frac{v^2}{4R^2 - v^2t^2} \tag{2.219}$$

To produce the interaction forces between the body 1 and the supports A and B one has to produce its free body diagram. It is shown in Fig.62Application of the

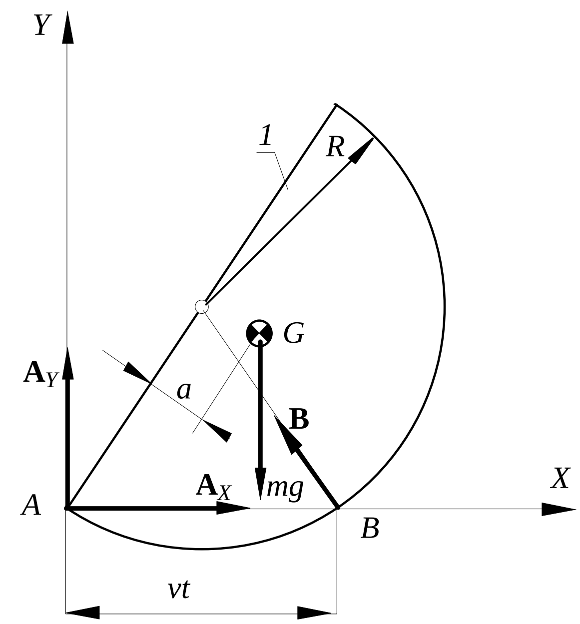


Figure 62

generalized Newton's equation to the body 1 yields

$$\begin{aligned} ma_X &= A_X - B\sin\alpha \\ ma_Y &= A_Y + B\cos\alpha - mg \\ I_A(-\ddot{\alpha}) &= -mg(R\sin\alpha + a\cos\alpha) + Bvt\cos\alpha \end{aligned} \tag{2.220}$$

Solving this equations with respect to the unknown A_X, A_Y and B one can get

$$\begin{aligned}
B &= \frac{mg(R\sin\alpha + a\cos\alpha) - I_A\ddot{\alpha}}{vt\cos\alpha} \\
A_X &= m(R\ddot{\alpha}\cos\alpha - R\dot{\alpha}^2\sin\alpha - a\ddot{\alpha}\sin\alpha - a\dot{\alpha}^2\cos\alpha) + \\
&\quad + \frac{mg(R\sin\alpha + a\cos\alpha) - I_A\ddot{\alpha}}{vt\cos\alpha}\sin\alpha \\
A_Y &= mg + m(-R\ddot{\alpha}\sin\alpha - R\dot{\alpha}^2\cos\alpha - a\ddot{\alpha}\cos\alpha + a\dot{\alpha}^2\sin\alpha) + \\
&\quad - \frac{mg(R\sin\alpha + a\cos\alpha) - I_A\ddot{\alpha}}{vt}
\end{aligned} \tag{2.221}$$

To produce expression for the driving force it is necessary to consider the equilibrium of the support B. Its free body diagram is given in Fig.63.

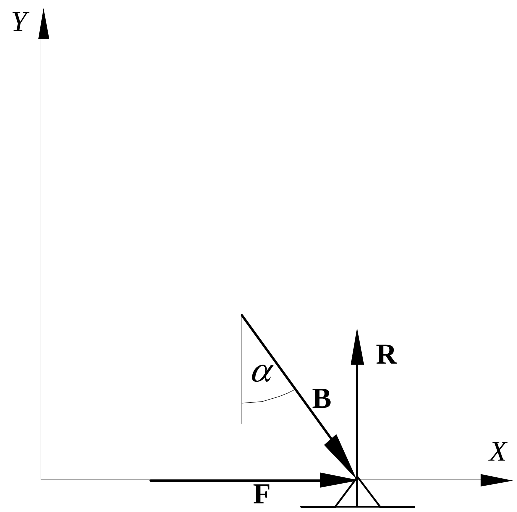


Figure 63

This support is in an equilibrium if

$$F + B\sin\alpha = 0 \tag{2.222}$$

Hence

$$F = -\frac{mg(R\sin\alpha + a\cos\alpha) - I_A\ddot{\alpha}}{vt\cos\alpha}\sin\alpha \tag{2.223}$$

where

$$\alpha = \arcsin\frac{vt}{2R} \tag{2.224}$$

Problem 40

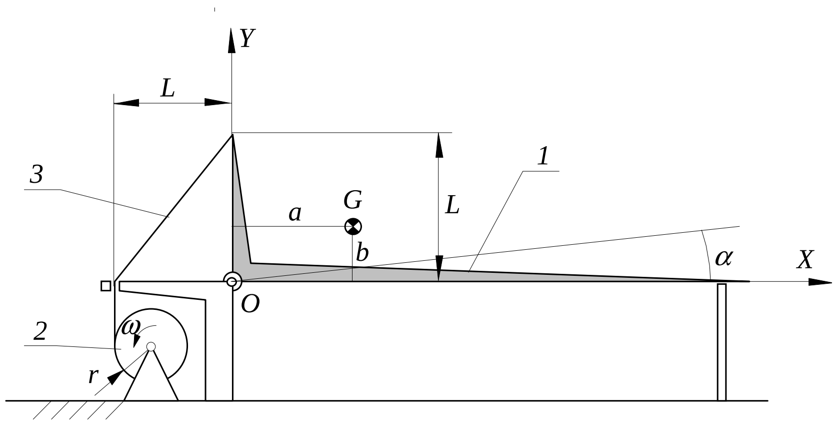


Figure 64

The street barrier *1* shown in Fig. 64 is being raised with help of the massless rope *3* and the winch *2*. The winch rotates with the constant angular velocity ω and the radius of its drum is r. The barrier can be treated as a rigid body of mass m and the moment of inertia I_O about the axis through its point of rotation O. The dimensions a and b locate position of its centre of gravity G.

Produce

1. the expression for the absolute angular velocity of the barrier

Answer:

$\alpha = 90^o - 2\arcsin\frac{1}{2}\left(\sqrt{2} - \frac{r}{L}\omega t\right)$ $\qquad \omega_1 = \dot{\alpha} = \frac{\frac{r}{L}\omega}{\sqrt{1-\frac{1}{4}(\sqrt{2}-\frac{r}{L}\omega t)^2}}$

2. the expression for the absolute velocity and the absolute acceleration of the centre of gravity of the barrier

Answer:

$\mathbf{v}_G = \mathbf{i}_1(-\omega_1 b) + \mathbf{j}_1(\omega_1 a)$ $\qquad \mathbf{a}_G = \mathbf{i}_1(-\dot{\omega}_1 b - a\omega_1^2) + \mathbf{j}_1(\dot{\omega}_1 a + b\omega_1^2)$

3. the expression for the kinetic energy of the barrier

Answer:

$T = \frac{1}{2} I_0 \omega_1^2$

4. the expression for the tension in the rope *3*

Answer:

$T = \frac{1}{L\cos\left(45^o - \frac{\alpha}{2}\right)}\left(\ddot{\alpha} I_O + mg\left(a\cos\alpha - b\sin\alpha\right)\right)$

Problem 41

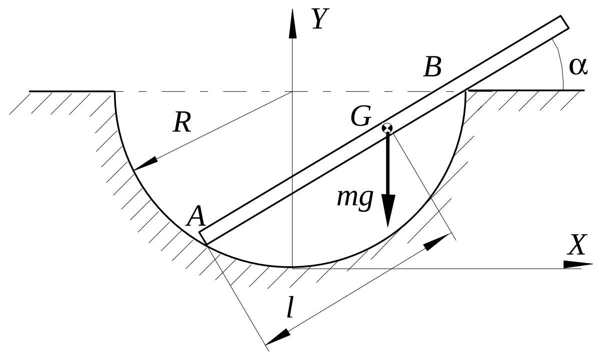


Figure 65

The rod of length $2l$ and mass m (see Fig. 65) can move in the vertical plane XY. It is supported by the edge B and its left hand side A is supported by the cylindrical surface of radius R. Produce the equations of motion of the rod and the expressions for the reaction force at A and B.

Answer:

$R_A = \frac{1}{\cos\alpha}(ma_{GX}\cos\alpha + ma_{GY}\sin\alpha + mg\sin\alpha)$

$R_B = \frac{1}{\cos\alpha}(ma_{GY}\cos 2\alpha - ma_{GX}\sin 2\alpha + mg\cos 2\alpha)$

$\ddot{\alpha}I + \frac{l\sin\alpha}{\cos\alpha}(ma_{GX}\cos\alpha + ma_{GY}\sin\alpha + mg\sin\alpha) - \frac{2R\cos\alpha - l}{\cos\alpha}(ma_{GY}\cos 2\alpha - ma_{GX}\sin 2\alpha + mg\cos 2\alpha) = 0$

Problem 42

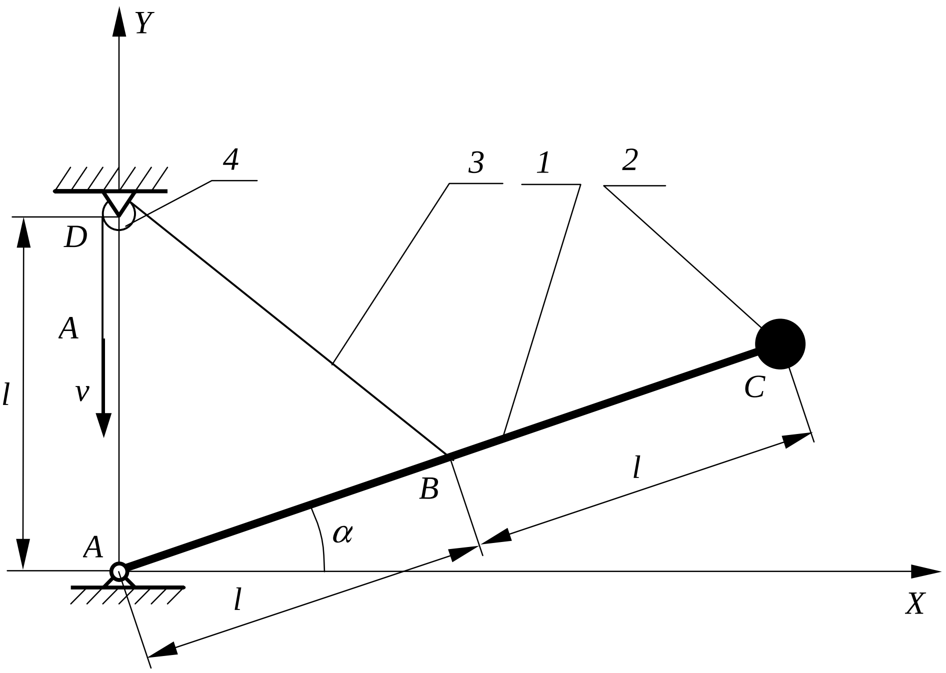


Figure 66

The link ABC is assembled of the uniform rod 1 of mass m_1 and the element 2 that can be treated as a particle of mass m_2. This link, of length $2l$, is being raised by means of the massless rope 3 and the pulley 4 with the constant velocity v. The radius of the pulley is negligible. The plane XY of this system of coordinates is vertical.

Produce:

1. the moment of inertia of link ABC about the axis through its centre of gravity G

Answer:

$I_G = \frac{m_1 l^2}{3} + m_1(c-l)^2 + m_2(2l-c)^2$

where

$c = \frac{m_1+2m_2}{m_1+m_2} l$

2. the moment of inertia of link ABC about the axis through the point of rotation A

Answer:

$I_A = I_G + (m_1+m_2)c^2$

3. the expression for the tension in the rope 3

Answer:

$T = \frac{I_A \ddot{\alpha} + (m_1+m_2)gc\cos\alpha}{l\cos(45^o - \frac{\alpha}{2})}$; where $\alpha = 90^o - 2\arcsin(\frac{\sqrt{2}}{2} - \frac{v}{2l}t)$

4. the expression for the components of the interaction force at the support A

Answer:

$R_X = -(m_1+m_2)c(\ddot{\alpha}\sin\alpha + \dot{\alpha}^2\cos\alpha) + T\cos(45^o - \frac{\alpha}{2})$

$R_Y = +(m_1+m_2)c(\ddot{\alpha}\cos\alpha + \dot{\alpha}^2\sin\alpha) - T\sin(45^o - \frac{\alpha}{2}) + (m_1+m_2)g$

Problem 43

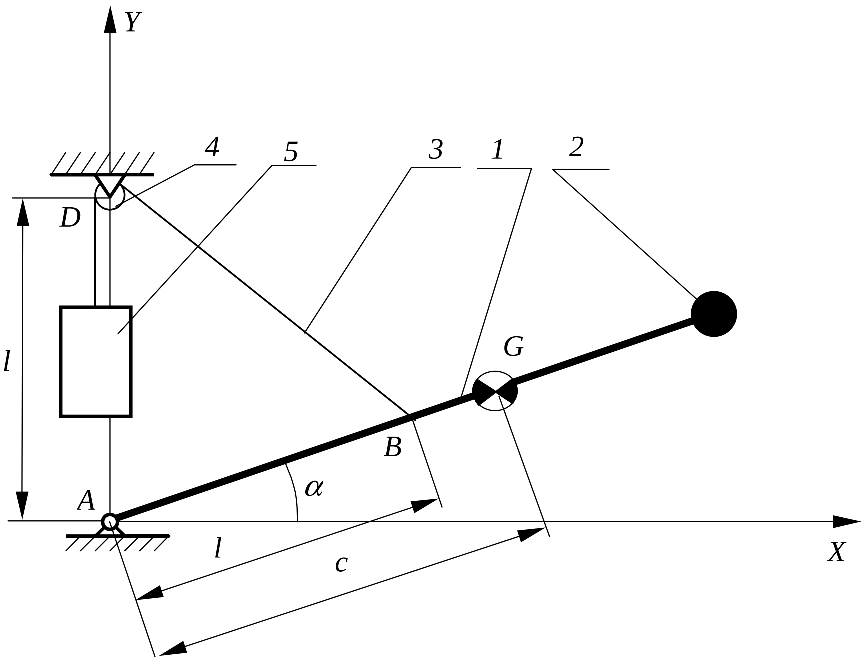


Figure 67

The link $1-2$ can be treated as a rigid body of mass m and moment of inertia I_G about the axis through its centre of rotation G. It is being raised by means of the massless rope 3, the pulley 4 and the block 5. The radius of the pulley is negligible and the mass of the block 5 is M. The system has one degree of freedom and the coordinate α determines its position with respect to the inertial system of coordinates XYZ. The plane XY of this system of coordinates is vertical.

Produce the differential equation of motion of the system.

Answer:

$I_A\ddot{\alpha} - M\ddot{Y}l\cos(45^o - \frac{\alpha}{2}) - Mgl\cos(45^o - \frac{\alpha}{2}) + mgc\cos\alpha = 0$

where

$Y = \left[1 - \sqrt{2} + 2\sin(45^o - \frac{\alpha}{2})\right] l$ and $I_A = I_G + mc^2$

Chapter 3
DYNAMICS OF PLANE MECHANISMS

In many applications we deal with a number of rigid bodies connected to each other in some manner. These connections, called *kinematic constrains* or *joints*, impose additional limitations on the relative motion of one body with respect to the other. Such a constrained set of rigid bodies forms *the mechanism.* In this chapter we will assume that all bodies (*links*) involved in the mechanism perform plane motion and that the plane of motion is the plane of symmetry of the system considered

3.1 CONSTRAINTS

The unconstrained rigid body j performing the plane motion has three degrees of freedom. Therefore, its relative position with respect to another body i can be uniquely determined by three independent coordinates. Usually the three coordinates x_i,y_i, α_i, are chosen as shown in Fig. 1 If the body is connected to another, the three coordi-

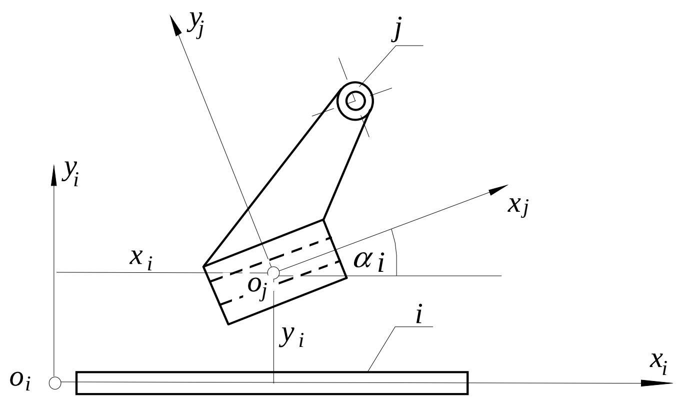


Figure 1

nates are not independent and we can produce a number of analytical relationships between them. These analytical relationships are called *constraint equations.* For example, if the two bodies i and j are connected as shown in Fig. 2, one may produce the following constraint equations

$$\begin{aligned} y_i &= 0 \\ \alpha_i &= 0 \end{aligned} \tag{3.1}$$

The number of constraint equations will be denoted by NCE. Since in the above example there is $NCE = 2$ constraint equations, only $3 - NCE = 1$ coordinates

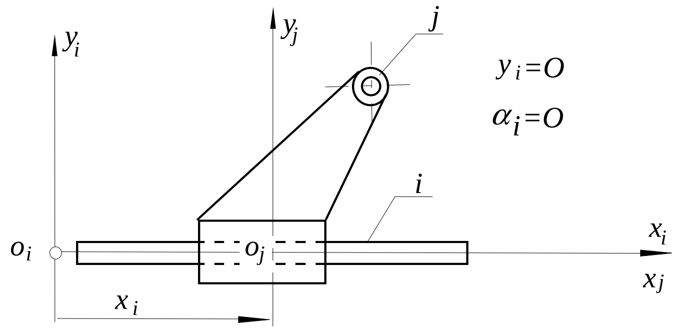


Figure 2

may be considered as independent. In the case considered, x_i may be chosen as the independent coordinate.

DEFINITION: The number of independent coordinates c which uniquely determine the relative position of two constrained bodies is called *class of constraint.*

Hence the class of a constraint is determined by formula

$$c = 3 - NCE \tag{3.2}$$

Figures 3 to 6 provide with more examples of possible constraints.

Fig. 3 presents constraint that allows the relative rotation only. Hence its class c is equal to 1. In this case the interaction force has two unknown scalar components. Number of these unknown components of the interaction force will be denoted by r.

A different form of the constrain of the class 1 is shown in Fig. 4. This constraint allows for translation only.

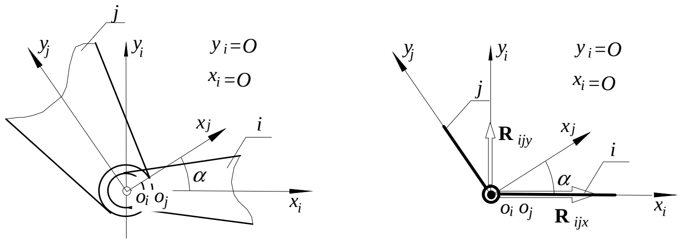
For the constraint shown in Fig. 5, if we assume that the rolling motion of the link i with respect to the link j is without slipping, one can formulate 2 equation between the 3 coordinates x_i, y_i and α_i. One of these equations must reflect the fact that there is always a contact point C and the other must guarantee that the length of the arc CD_j is equal to the arc CD_i.

$$\overset{\frown}{CD}_j = \overset{\frown}{CD}_i \tag{3.3}$$

Hence the class of this constraint is equal to 1 ($c = 1$) and the number of scalar components of the interaction force $r = 2$.

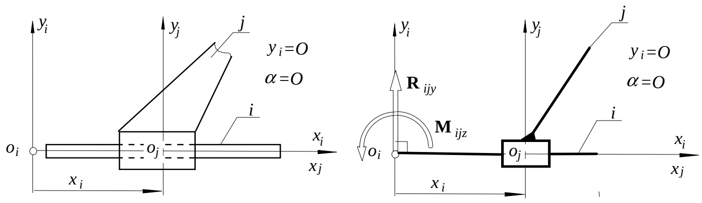
Situation when slipping is allowed between link i and j is illustrated in Fig. 6. In this case the constraint equation 3.3 does not have to be fulfilled and therefore we can produce only one constraint equation. Hence, the class of this constraint is equal to 2 and there is only one unknown component of interaction force ($r = 1$).By inspection of those diagrams one can notice that the number of the unknown components of interaction forces r is

$$r = 3 - c = 3 - (3 - NCE) = NCE \tag{3.4}$$



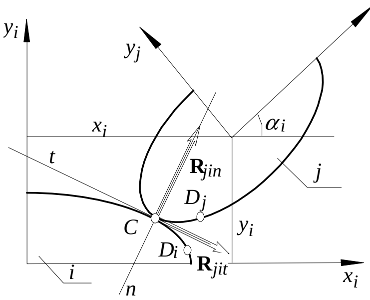
NCE=2 c=1 r=2

Figure 3



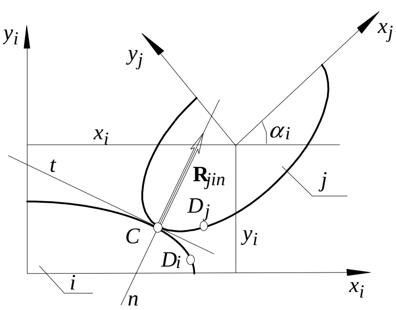
NCE=2 c=1 r=2

Figure 4



NCE=2 c=1 r=2

Figure 5



NCE=1 c=2 r=1

Figure 6

3.2 MOBILITY - GENERALIZED COORDINATES.

Let us consider a mechanism which is assembled of n links connected to each other by means of p constraints. Among the p constraints there are p_1 constraints of class 1 and p_2 constraints of class 2. Evidently,

$$p = p_1 + p_2 \tag{3.5}$$

Unconnected links of the mechanism would have $3n$ degree of freedom. Therefore to determine uniquely position of the disintegrated mechanism one has to use $3n$ independent coordinates. Since each constraint of class c takes away from the mechanism considered $3 - c$ degree of freedom, the number of degree of freedom which is left after imposition of the p constraints is

$$\begin{aligned} MO &= 3n - (3-1)p_1 - (3-2)p_2 \\ &= 3n - 2p_1 - p_2 \end{aligned} \tag{3.6}$$

DEFINITION: The number MO determined by the formula 3.6 is called *mobility* and represents number of coordinates one has to use to determine position of the system with respect to the inertial space.

Mobility $MO \leqslant 0$ corresponds to a kinematic chain that can not move. A mechanism must have mobility $MO > 0$.

DEFINITION: Coordinates which uniquely determine position of a mechanism with respect to the inertial space are called *generalized coordinates.*

As an example let us consider mechanism presented in Fig. 7. Constraint K is the only constraint of class *2.* Hence ($p_2 = 1$). The other constraints $A, B, C, D, E, F, G, H, J$ are of class *1* ($p_1 = 9$). Since there is $n = 7$ moving links, according to Eq. 3.6 mobility of the mechanism is

$$MO = 3 \cdot 7 - 2 \cdot 9 - 1 \cdot 1 = 2 \tag{3.7}$$

Hence, one has to introduce $MO = 2$ independent coordinates. One of them, e.g. α (see Fig.. 7) can be chosen arbitrarily. It is easy to see that in the case considered we deal with a local degree of freedom. Namely, the roller 7 can freely rotates and this rolling motion has no influence on motion of the other links. This local degree of freedom has been deducted by the formula 3.7 and the coordinate β which determine this motion must be chosen as the second one. Such a local degree of freedom should be eliminated.

Figure 8 shows the equivalent mechanism without the local degree of freedom.Mobility of the modified mechanism is

$$MO = 3 \cdot 6 - 8 \cdot 2 - 1 \cdot 1 = 1 \tag{3.8}$$

Therefore an unique position of the mechanism shown in Fig. 8 can be determine by one coordinate α only.

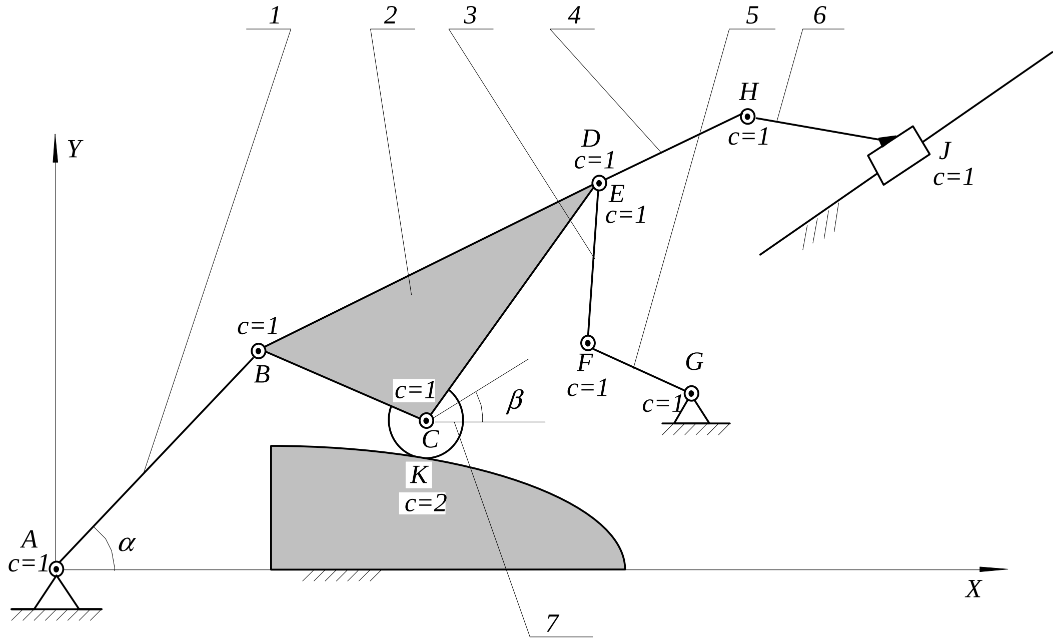


Figure 7

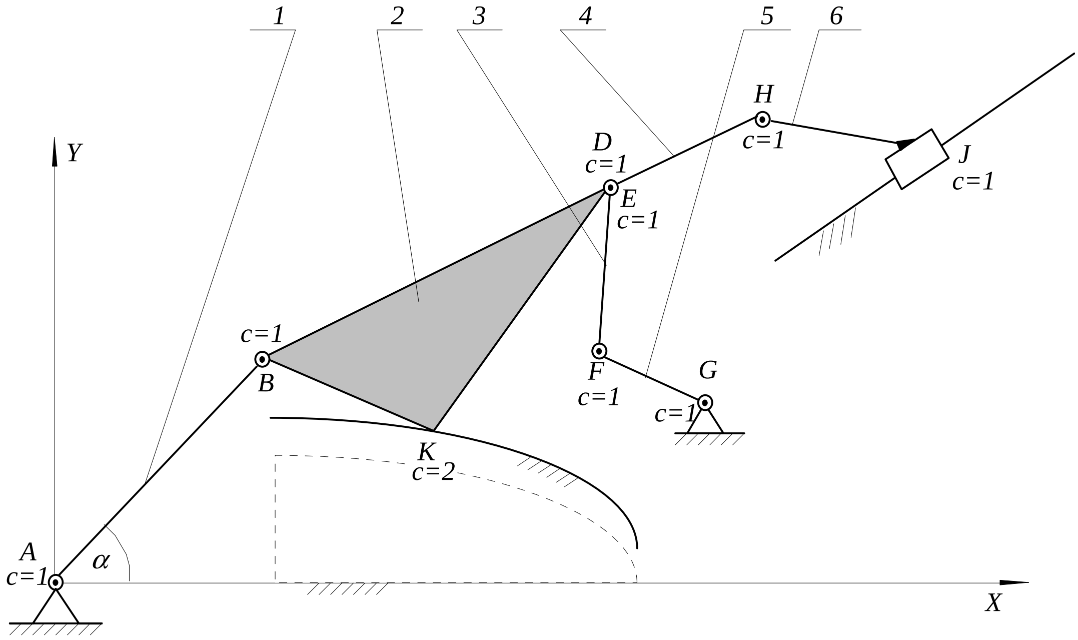


Figure 8

In a general case a set of MO coordinates q_1, q_2, q_3..... q_{MO} determines position of the mechanism considered. Hence it is always possible to express the position vector of the arbitrarily chosen point as a function of this set of coordinates.

$$\mathbf{r} = \mathbf{r}(q_1, q_2, ...q_{MO}) \tag{3.9}$$

3.3 NUMBER OF DEGREE OF FREEDOM - DRIVING FORCES.

In some practical applications, we need to assume that motion along some of the generalized coordinates is given as an explicit function of time. Let $L < MO$ be the number of coordinates along which motion of the mechanism is known. In this case the number of degree of freedom of the system is reduced by L. Hence, the actual number of degree of freedom is

$$M = MO - L \tag{3.10}$$

DEFINITION: *Number of degree of freedom* **is the number of independent generalized coordinates that one has to use to determine uniquely position of a mechanism with respect to the inertial space.**

Since motion along L coordinates has been assumed, one has to introduce L independent forces which guarantee the assumed motion along the L coordinates. These unknown forces are called *driving forces*. In the considered case, each position vector can be expressed by M generalized coordinates and time t.

$$\mathbf{r} = \mathbf{r}(q_1, q_2, q_M, t) \tag{3.11}$$

DEFINITION: If all possible points of a mechanism have positions vector of form 3.9, the system is called *scleronomic.*

DEFINITION: If at least one point of a mechanism has position vector of form 3.11, the system is called *rheonomic.*

3.3.1 *Problems*

Problem 44

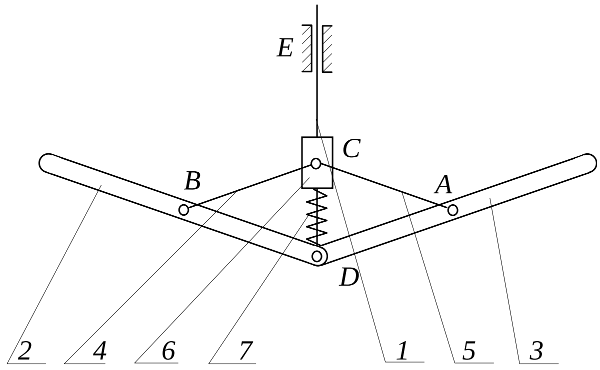


Figure 9

Calculate mobility of the mechanism shown in Fig. 9

Solution

The mechanism is assembled of $n = 6$ moving links. There are 2 constraints of class 1 at D, 3 constraints of class 1 at C and 1 constraints of class 1 at A, B, and E. Hence, the number of all the constraints of class 1 is $p_1 = 8$.Introduction of these data into the formula for mobility yields

$$MO = 3n - 2p_1 - p_2 = 3 \cdot 6 - 2 \cdot 8 = 2 \tag{3.12}$$

Problem 45

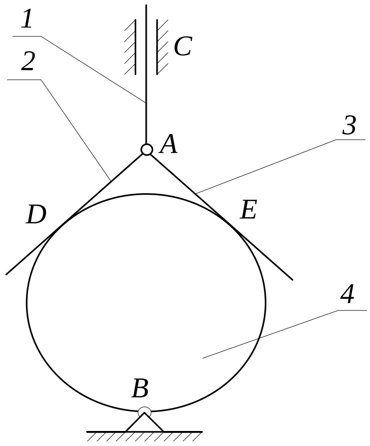


Figure 10

Calculate mobility of the mechanism shown in Fig. 10

Solution

The mechanism is assembled of $n = 4$ moving links. There are 4 constraints of class 1 at A, B, and C (at A there are 2 constraints). At D and E there are 2 constraints of class 2. Hence

$$MO = 3n - 2p_1 - p_2 = 3 \cdot 4 - 2 \cdot 4 - 2 = 2 \tag{3.13}$$

Problem 46

Calculate mobility of the mechanism shown in figures 11,12 and 13.

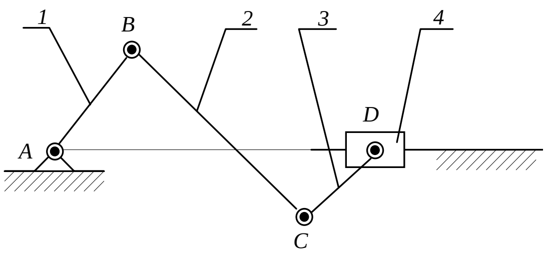


Figure 11

Answer:
$MO = 2$

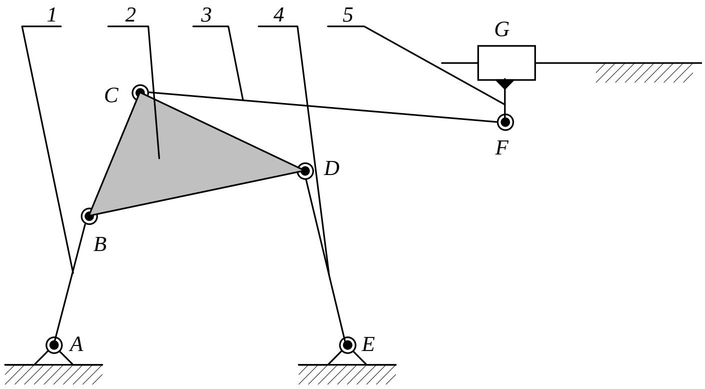


Figure 12

Answer:
$MO = 1$

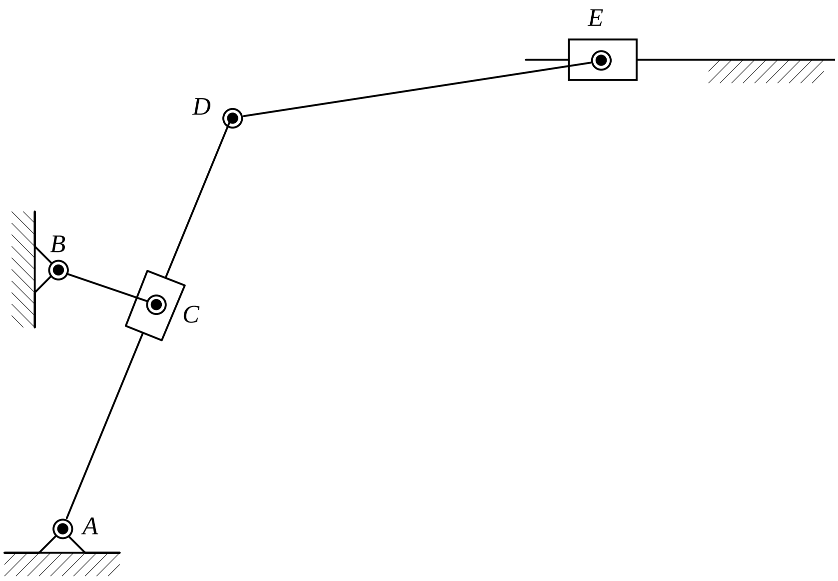


Figure 13

Answer:
$MO = 1$

3.4 EQUATIONS OF MOTION.

The discusion carried out in the previous section leads to conclusion that for any mechanism assembled of n rigid bodies we deal with

M — unknown functions representing a motion along M generalized coordinates q_m,

L — unknown driving forces F_d,

$r = 2p_1 + p_2$ — number of unknown scalar components at $p = p_1 + p_2$ constraints.

Hence, the total number of all unknowns is

$$\begin{aligned} TNU &= M + L + 2p_1 + p_2 \\ &= MO + 2p_1 + p_2 \end{aligned} \tag{3.14}$$

Upon introducing Eq. 3.6 into Eq. 3.14) we have

$$TNU = 3n - 2p_1 - p_2 + 2p_1 + p_2 = 3n \tag{3.15}$$

On the other hand one can split the mechanism into individual links. Then, for each of the n moving links it is possible to produce 3 scalar equations of motion of the following form.

$$\begin{aligned} m_j \mathbf{a}_j &= \mathbf{F}_j \\ I_j \varepsilon_j &= M_j \end{aligned} \tag{3.16}$$

An example of the free body diagram of such a $j - th$ link is given in Fig. 14.

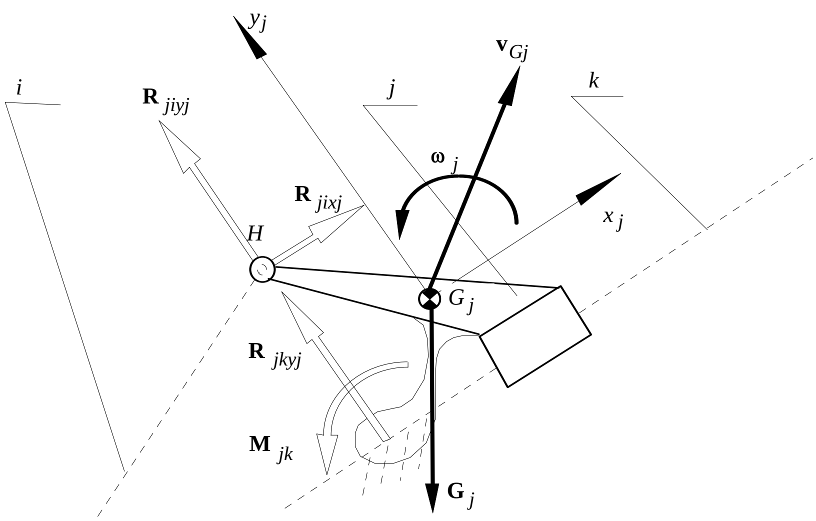


Figure 14

If the absolute velocity of its centre of gravity is resolved along the body system of coordinates,

$$\mathbf{v}_{Gj} = \dot{\mathbf{r}}_{Gj} = \mathbf{i}_j(v_{Gxj}) + \mathbf{j}_j(v_{Gyj}) \tag{3.17}$$

its acceleration $\mathbf{a}_{Gj}$ is

$$\begin{aligned}
\mathbf{a}_{Gj} &= \ddot{\mathbf{r}}_{Gj} = \mathbf{v}'_{Gj} + \boldsymbol{\omega}_j \times \mathbf{v}_{Gj} = \mathbf{i}_j(\dot{v}_{Gxj}) + \mathbf{j}_j(\dot{v}_{Gyj}) + \begin{vmatrix} \mathbf{i}_j & \mathbf{j}_j & \mathbf{k}_j \\ 0 & 0 & \omega_j \\ v_{Gxj} & v_{Gyj} & 0 \end{vmatrix} = \\
&= \mathbf{i}_j(\dot{v}_{Gxj} - v_{Gyj}\omega_j) + \mathbf{j}_j(\dot{v}_{Gyj} + v_{Gxj}\omega_j) \qquad (3.18)
\end{aligned}$$

Introduction of the above expression into the generalized Newton's equations 3.16 yields

$$\begin{aligned}
m_j(\dot{v}_{Gjxj} - v_{Gjyj}\omega_j) &= F_{jxj} \\
m_j(\dot{v}_{Gjyj} + v_{Gjxj}\omega_j) &= F_{jyj} \\
I_{jzGj}\dot{\omega}_j &= M_{jzj} \qquad (3.19)
\end{aligned}$$

Kinematics (Section. 2.1.) provides methods which permit the kinematic parameters $\mathbf{v}_G$ and $\boldsymbol{\omega}$ to be expressed as functions of the M unknown generalized coordinates q_m and time t.

$$\begin{aligned}
\mathbf{v}_G &= \mathbf{v}_G(q_1...q_m....q_M, \dot{q}_1...\dot{q}_m....\dot{q}_M, t) \\
\boldsymbol{\omega} &= \boldsymbol{\omega}(q_1...q_m....q_M, \dot{q}_1...\dot{q}_m....\dot{q}_M, t) \qquad (3.20)
\end{aligned}$$

Since the system considered is assembled of n bodies, we are able to generate $3n$ equations with $3n$ unknown. These equations are linear with respect to the $3n - M$ unknown components of interaction forces and driving forces. Therefore always it is possible to eliminate them and produce M differential equations known as *differential equation of motion of a mechanism.* Their solution presents motion of the system considered. Upon introducing of this solution into the remaining equations one can obtain expressions for all reactions and driving forces.

3.4.1 Problems

Problem 47

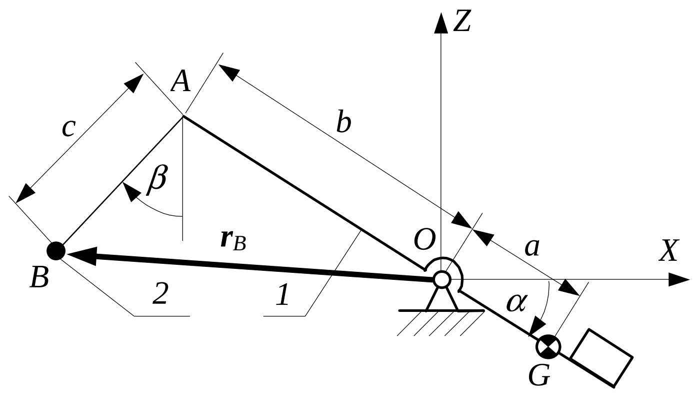


Figure 15

The physical model of a catapult is presented in Fig. 15. The arm 1 can be considered as a rigid body of the mass M and the moment of inertia about axis through its centre of gravity I_G. It rotates about the horizontal axis Y of the inertial system of coordinates XYZ. Its instantaneous position is determined by the angular displacement α. The projectile 2 that can be considered as a particle of the mass m is attached to the arm by means of the massless rope AB. The instantaneous position of the projectile is determined by the angular displacement β.

Produce:

1. the expression for the moment of inertia of the arm 1 about axis through the point O
2. the expression for the components of the absolute velocity and the absolute acceleration of the particle B along the system of coordinates XYZ
3. the expression for the kinetic energy function of the system
4. the equations of motion of the system

Solution

Moment of inertia of the arm 1 about the axis of rotation O

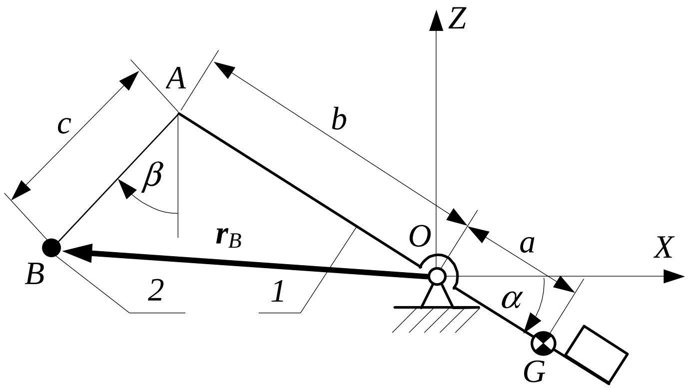


Figure 16

$$I_O = I_G + Ma^2 \tag{3.21}$$

The absolute velocity and the absolute acceleration of the particle B

According to the diagram shown in Fig. 16, the absolute position vector of the point B is

$$\mathbf{r}_B = \mathbf{I}(-b\cos\alpha - c\sin\beta) + \mathbf{K}(b\sin\alpha - c\cos\beta) \tag{3.22}$$

Differentiation of this vector with respect to time yields the vector of the absolute velocity and the absolute acceleration respectively.

$$\mathbf{v}_B = \dot{\mathbf{r}}_B = \mathbf{I}(b\dot{\alpha}\sin\alpha - c\dot{\beta}\cos\beta) + \mathbf{K}(b\dot{\alpha}\cos\alpha + c\dot{\beta}\sin\beta) \tag{3.23}$$

$$\begin{aligned}\mathbf{a}_B &= \ddot{\mathbf{r}}_B = \mathbf{I}a_X + \mathbf{K}a_Z = \\ &= \mathbf{I}(b\ddot{\alpha}\sin\alpha + b\dot{\alpha}^2\cos\alpha - c\ddot{\beta}\cos\beta + c\dot{\beta}^2\sin\beta) + \\ &\quad +\mathbf{K}(b\ddot{\alpha}\cos\alpha - b\dot{\alpha}^2\sin\alpha + c\ddot{\beta}\sin\beta + c\dot{\beta}^2\cos\beta)\end{aligned} \tag{3.24}$$

The kinetic energy of the system

$$T = \tfrac{1}{2}I_O\dot{\alpha}^2 + \tfrac{1}{2}m\left[(b\dot{\alpha}\sin\alpha - c\dot{\beta}\cos\beta)^2 + (b\dot{\alpha}\cos\alpha + c\dot{\beta}\sin\beta)^2\right]$$

Equations of motion of the system

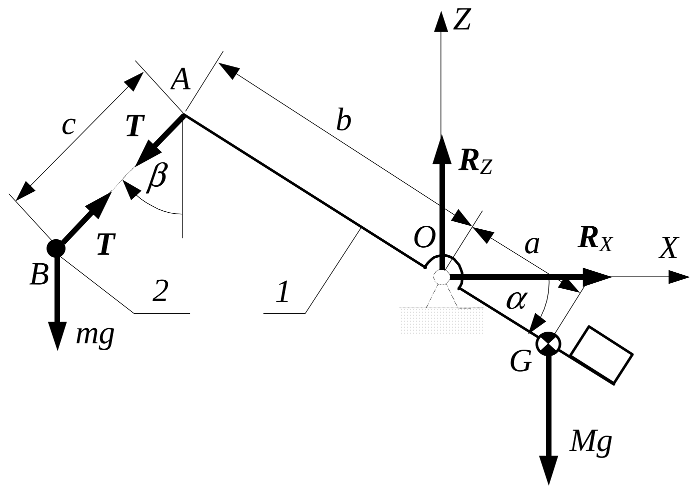


Figure 17

The free body diagram of the particle B and the arm 1 are shown in Fig. 17. Application of the Newton's law to the particle B yields

$$ma_X = T\sin\beta \tag{3.25}$$
$$ma_Z = T\cos\beta - mg \tag{3.26}$$

Application of the generalized Newton's law to the arm yields

$$I_O\ddot{\alpha} = Mga\cos\alpha - Tb\cos(\beta-\alpha) \tag{3.27}$$

Elimination of the tension T from these equations results in the following equations of motion of the system along the generalized coordinates α and β

$$\begin{aligned} ma_Z - ma_X\frac{\cos\beta}{\sin\beta} + mg &= 0 \\ I_O\ddot{\alpha} - Mga\cos\alpha + ma_X b\cos(\beta-\alpha)\frac{1}{\sin\beta} &= 0 \end{aligned} \tag{3.28}$$

where

$$\begin{aligned} a_X &= b\ddot{\alpha}\sin\alpha + b\dot{\alpha}^2\cos\alpha - c\ddot{\beta}\cos\beta + c\dot{\beta}^2\sin\beta \\ a_Z &= b\ddot{\alpha}\cos\alpha - b\dot{\alpha}^2\sin\alpha + c\ddot{\beta}\sin\beta + c\dot{\beta}^2\cos\beta \end{aligned} \tag{3.29}$$

Problem 48

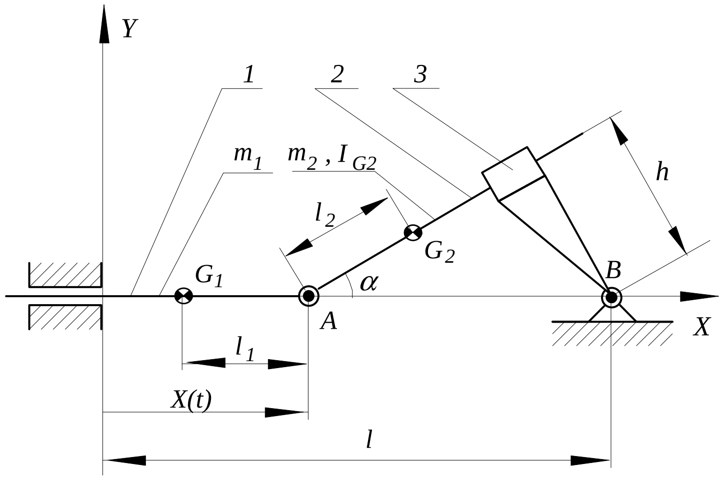


Figure 18

The link 1 of the mechanism shown in Fig. 18 is moving along the axis X of the inertial system of coordinates XYZ. Its motion is determined by the function $X(t)$. Mass of the link 1 is m_1 and its centre of mass is located at the point G_1. The link 2 has mass m_2 and the moment of inertia I_{G2} about the axis through its centre of gravity G_2. Produce expression for the reactions at all kinematic constraints as well as the expression for the driving force that must be apply to the link 1.

Solution

To take advantage of the Newton's equations it is necessary to developed expression for the angular acceleration of all heavy links and the linear acceleration of their centres of gravity. To this end let us consider the vector polygon shown in Fig. 19.

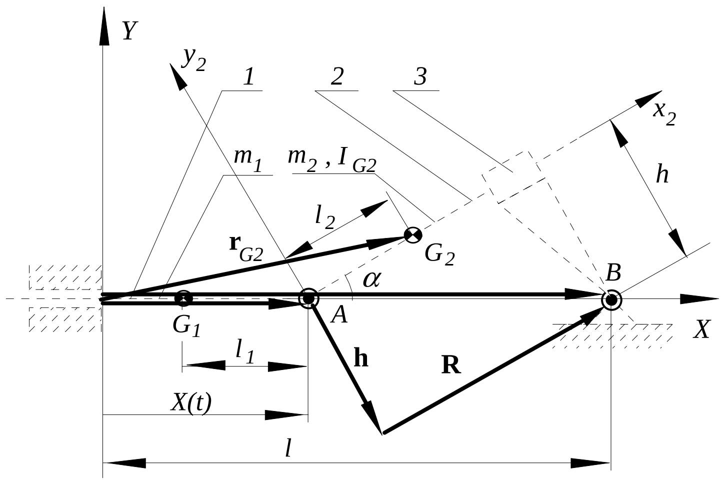


Figure 19

$$\mathbf{I}X(t) - \mathbf{j}_2 h + \mathbf{i}_2 R = \mathbf{I}l \tag{3.30}$$

This equation is equivalent to two scalar equations that can be obtained by subsequent multiplication of this equation by the unit vectors associated with any system of coordinates. Therefore, multiplying the equation 3.30 by **I** and **J** we have

$$\begin{aligned} X(t) + h\sin\alpha + R\cos\alpha &= l \\ -h\cos\alpha + R\sin\alpha &= 0 \end{aligned} \tag{3.31}$$

In this case it is easy to obtained the analytical solution of these two equations with respect to the unknown α and R.

$$\begin{aligned} \alpha &= \arcsin\frac{h}{l-X} \\ R &= \sqrt{(l-X)^2 - h^2} \end{aligned} \tag{3.32}$$

The angular velocity of the link 1 is

$$\omega_1 = 0 \tag{3.33}$$

The angular velocity of the link 2 is equal to the velocity of its body system of coordinates $x_2 y_2$.

$$\omega_2 = \mathbf{K}\dot{\alpha} \tag{3.34}$$

Since there is no angular relative motion between the link 2 and 3, the absolute angular velocity of the link 3 is

$$\omega_3 = \omega_2 = \mathbf{K}\dot{\alpha} \tag{3.35}$$

Differentiation of the above expressions produces the angular accelerations.

$$\begin{aligned} \varepsilon_1 &= 0 \\ \omega_3 &= \omega_2 = \mathbf{K}\ddot{\alpha} \end{aligned} \tag{3.36}$$

The linear velocity of the link 1 is

$$\mathbf{v}_{G1} = \mathbf{I}\dot{X} \tag{3.37}$$

Its acceleration is

$$\mathbf{a}_{G1} = \mathbf{I}\ddot{X} \tag{3.38}$$

The linear velocity of the centre of gravity G_2 can be obtained by differentiation of the absolute position vector $\mathbf{r}_{G2}$ shown in Fig. 19.

$$\mathbf{r}_{G2} = \mathbf{I}X + \mathbf{i}_2 l_2 = \mathbf{i}_2\left(X\cos\alpha + l_2\right) + \mathbf{j}_2\left(-X\sin\alpha\right) \tag{3.39}$$

Hence

$$\begin{aligned} \mathbf{v}_{G2} &= \dot{\mathbf{r}}_{G2} = \mathbf{r}'_{G2} + \boldsymbol{\omega}_2 \times \mathbf{r}_{G2} \\ &= \mathbf{i}_2\left(\dot{X}\cos\alpha - X\dot{\alpha}\sin\alpha\right) + \mathbf{j}_2\left(-\dot{X}\sin\alpha - X\dot{\alpha}\cos\alpha\right) + \\ &\quad + \begin{vmatrix} \mathbf{i}_2 & \mathbf{j}_2 & \mathbf{k}_2 \\ 0 & 0 & \dot{\alpha} \\ X\cos\alpha + l_2 & -X\sin\alpha & 0 \end{vmatrix} \\ &= \mathbf{i}_2\left(\dot{X}\cos\alpha\right) + \mathbf{j}_2\left(-\dot{X}\sin\alpha + \dot{\alpha}l_2\right) \end{aligned} \tag{3.40}$$

The second differentiation yields the absolute linear acceleration

$$\begin{aligned} \mathbf{a}_{G2} &= \dot{\mathbf{v}}_{G2} = \mathbf{v}'_{G2} + \boldsymbol{\omega}_2 \times \mathbf{v}_{G2} \\ &= \mathbf{i}_2\left(\ddot{X}\cos\alpha - \dot{X}\dot{\alpha}\sin\alpha\right) + \mathbf{j}_2\left(-\ddot{X}\sin\alpha - \dot{X}\dot{\alpha}\cos\alpha + \ddot{\alpha}l_2\right) \\ &\quad + \begin{vmatrix} \mathbf{i}_2 & \mathbf{j}_2 & \mathbf{k}_2 \\ 0 & 0 & \dot{\alpha} \\ \dot{X}\cos\alpha & -\dot{X}\sin\alpha + \dot{\alpha}l_2 & 0 \end{vmatrix} \\ &= \mathbf{i}_2\left(\ddot{X}\cos\alpha - \dot{\alpha}^2 l_2\right) + \mathbf{j}_2\left(-\ddot{X}\sin\alpha + \ddot{\alpha}l_2\right) \end{aligned} \tag{3.41}$$

Fig. 20 shows the free body diagram for the link 1. Application of the Newton's generalized equations

$$\begin{aligned} \mathbf{a}m &= \mathbf{F} \\ \varepsilon I &= M \end{aligned} \tag{3.42}$$

to the link 1 yields the following three scalar equations

$$\begin{aligned} m_1\ddot{X} &= R_{12x2}\cos\alpha - R_{12y2}\sin\alpha + F_d \\ 0 &= -m_1 g + R_{10} + R_{12x2}\sin\alpha + R_{12y2}\cos\alpha \\ 0 &= M_{10} + R_{10}l_1 + R_{12y2}l_1\cos\alpha + R_{12x2}l_1\sin\alpha \end{aligned} \tag{3.43}$$

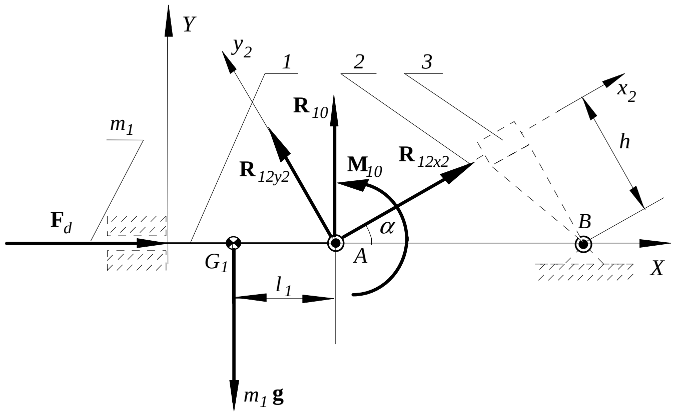


Figure 20

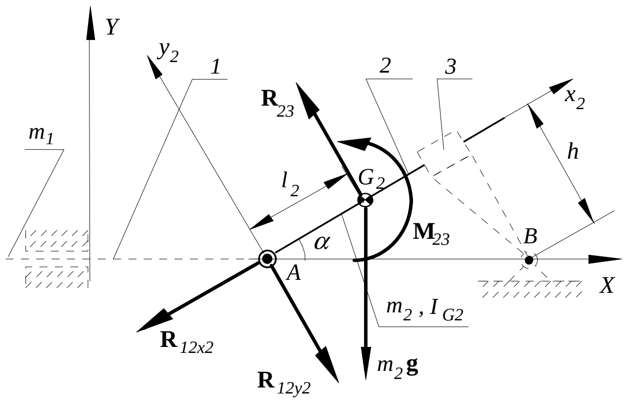


Figure 21

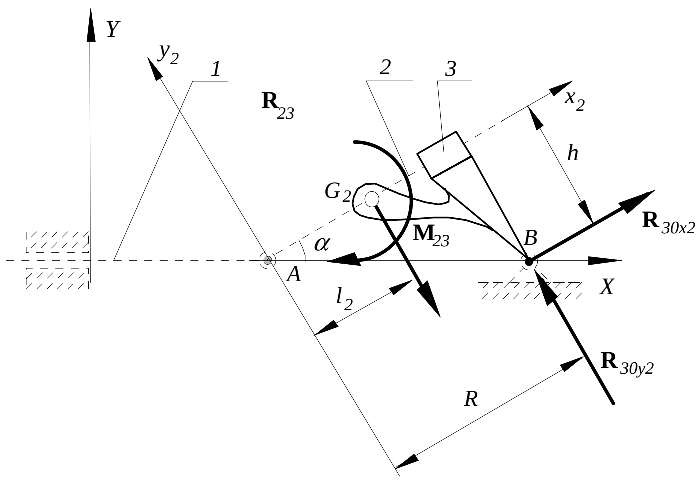


Figure 22

Fig. 21 shows the free body diagram for the link 2. Application of the Newton's generalized equations to the link 2 yields the following three scalar equations

$$\begin{aligned} m_2\left(\ddot{X}\cos\alpha - \dot{\alpha}^2 l_2\right) &= -R_{12x2} - m_2 g\sin\alpha \\ m_2\left(-\ddot{X}\sin\alpha + \ddot{\alpha} l_2\right) &= -R_{12y2} - m_2 g\cos\alpha + R_{23} \\ I_{G2}\ddot{\alpha} &= M_{23} + R_{12y2} l_1 \end{aligned} \tag{3.44}$$

Fig. 22 shows the free body diagram for the link 3. Application of the Newton's generalized equations to the link 3 yields the following three scalar equations

$$\begin{aligned} 0 &= R_{30x2} \\ 0 &= R_{30y2} - R_{23} \\ 0 &= -M_{23} + R_{30y2}\left(R - l_2\right) + R_{30x2} h \end{aligned} \tag{3.45}$$

The equations 3.43, 3.44 and 3.45 forms the non-homogeneous set of equations that is linear with respect to the unknown interaction forces R_{12x2}, R_{12y2}, R_{10}, M_{10}, R_{23}, M_{23} R_{30x2}, R_{30y2} and the driving force F_d.

$$\begin{aligned} R_{12x2}\cos\alpha - R_{12y2}\sin\alpha + F_d &= m_1\ddot{X} \\ R_{10} + R_{12x2}\sin\alpha + R_{12y2}\cos\alpha &= m_1 g \\ M_{10} + R_{10} l_1 + R_{12y2} l_1\cos\alpha + R_{12x2} l_1 \sin\alpha &= 0 \\ -R_{12x2} &= m_2\left(\ddot{X}\cos\alpha - \dot{\alpha}^2 l_2\right) + m_2 g\sin\alpha \\ R_{12y2} + R_{23} &= m_2\left(-\ddot{X}\sin\alpha - \ddot{\alpha} l_2\right) + m_2 g\cos\alpha \\ M_{23} + R_{12y2} l_1 &= I_{G2}\ddot{\alpha} \\ R_{30x2} &= 0 \\ R_{30y2} - R_{23} &= 0 \\ -M_{23} + R_{30y2}\left(R - l_2\right) + R_{30x2} h &= 0 \end{aligned} \tag{3.46}$$

These equations can be arranged in the following matrix form

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \sin\alpha & \cos\alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ l_1\sin\alpha & l_1\cos\alpha & l_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & l_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & h & R - l_2 & 0 \end{bmatrix} \begin{bmatrix} R_{12x2} \\ R_{12y2} \\ R_{10} \\ M_{10} \\ R_{23} \\ M_{23} \\ R_{30x2} \\ R_{30y2} \\ F_d \end{bmatrix} =$$

$$= \begin{bmatrix} m_1\ddot{X} \\ m_1 g \\ 0 \\ m_2\left(\ddot{X}\cos\alpha - \dot{\alpha}^2 l_2\right) + m_2 g\sin\alpha \\ m_2\left(-\ddot{X}\sin\alpha - \ddot{\alpha} l_2\right) + m_2 g\cos\alpha \\ I_{G2}\ddot{\alpha} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.47}$$

Hence the interaction forces and the driving force are

$$\begin{bmatrix} R_{12x2} \\ R_{12y2} \\ R_{10} \\ M_{10} \\ R_{23} \\ M_{23} \\ R_{30x2} \\ R_{30y2} \\ F_d \end{bmatrix} =$$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \sin\alpha & \cos\alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ l_1\sin\alpha & l_1\cos\alpha & l_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & l_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & h & R-l_2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} m_1\ddot{X} \\ m_1 g \\ 0 \\ m_2\left(\ddot{X}\cos\alpha - \dot{\alpha}^2 l_2\right) + m_2 g\sin\alpha \\ m_2\left(-\ddot{X}\sin\alpha - \ddot{\alpha} l_2\right) + m_2 g\cos\alpha \\ I_{G2}\ddot{\alpha} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.48}$$

where according to 3.32

$$\begin{aligned} \alpha &= \arcsin\frac{h}{l-X} \\ R &= \sqrt{(l-X)^2 - h^2} \end{aligned}$$

Problem 49

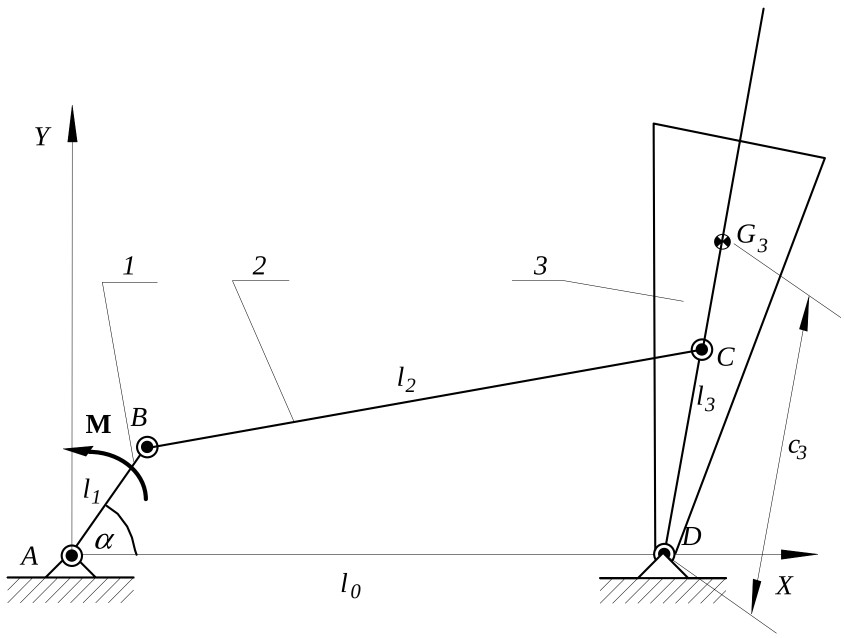


Figure 23

The mechanism shown in Fig. 23 is driven by the constant moment **M** applied to the link 1. The link 1 and 2 can be considered as massless. The link 3 has mass equal to m and its moment of inertia about D is I_D. Produce the differential equation of motion of the mechanism.

Solution

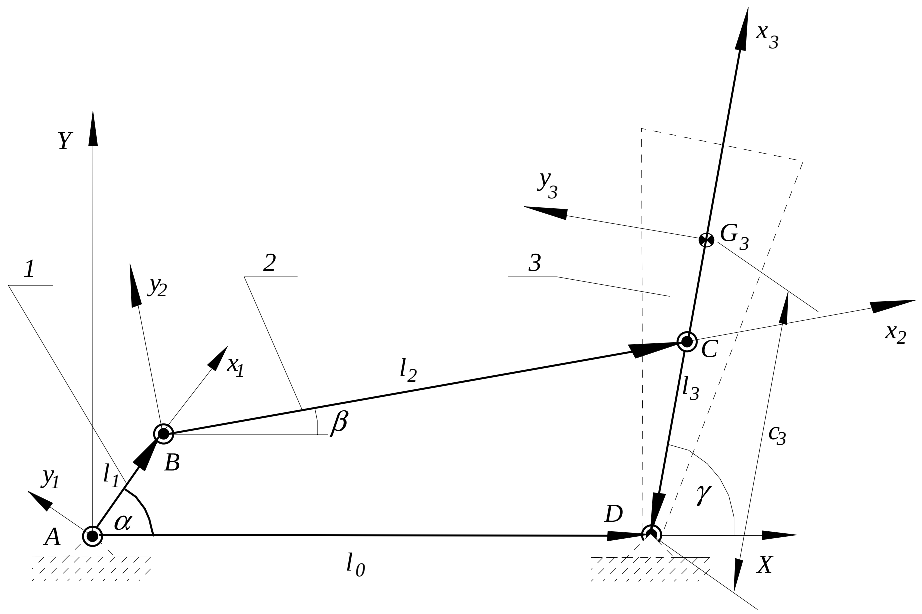


Figure 24

The mechanism has one degree of freedom and the angular displacement α is chosen as the generalized coordinate. The angular displacements β and γ (see Fig. 24) that determined position of the mechanism are functions of the independent coordinate α. These functions can be obtained by solving the following vectorial equation.

$$\mathbf{l}_1 + \mathbf{l}_2 + \mathbf{l}_3 = \mathbf{l}_0 \tag{3.49}$$

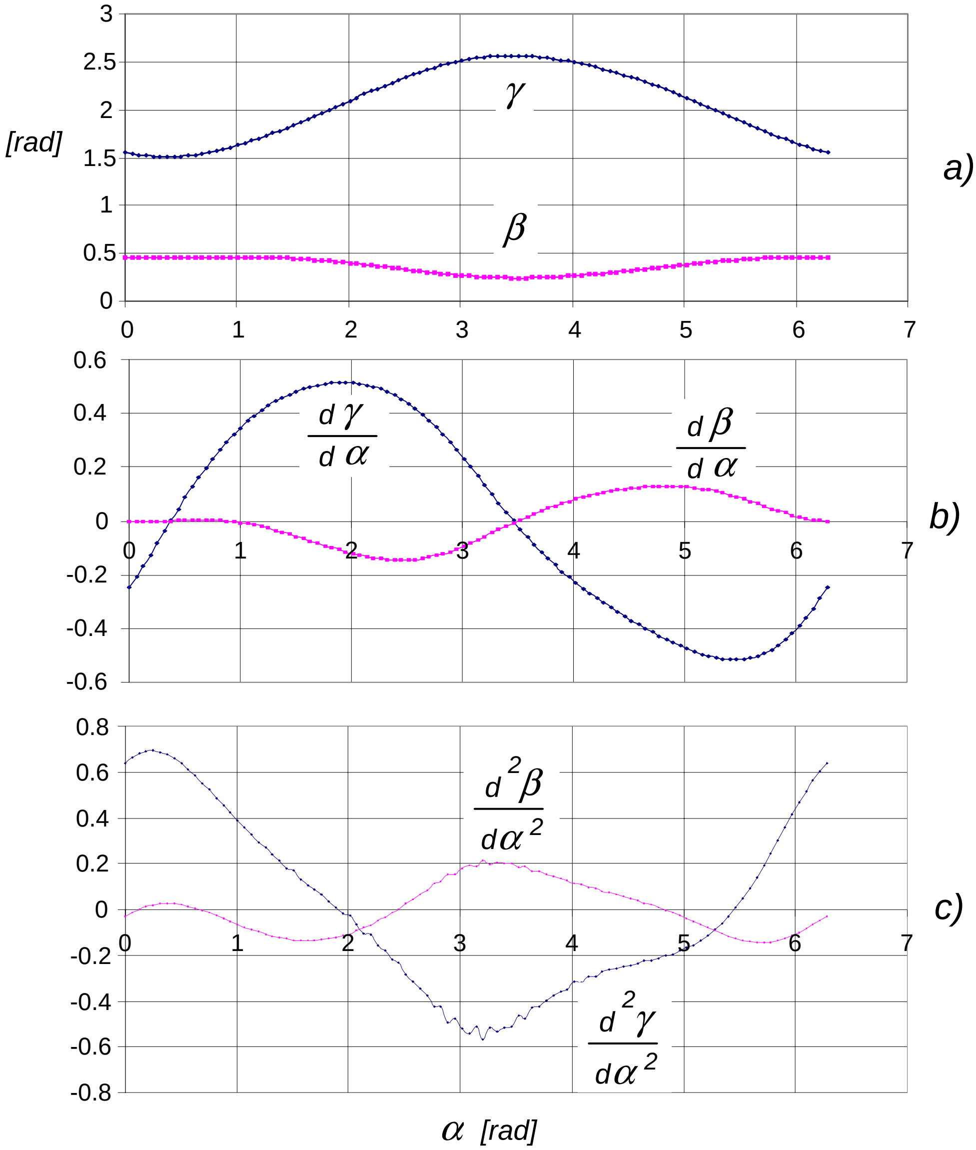
This equation is equivalent to the two scalar equations that one can get by multiplying it by the unit vectors **I** and **J** subsequently.

$$\begin{aligned} l_1 \cos\alpha + l_2 \cos\beta - l_3 \cos\gamma &= l_0 \\ l_1 \sin\alpha + l_2 \sin\beta - l_3 \sin\gamma &= 0 \end{aligned} \tag{3.50}$$

The above set of the nonlinear equations can not be solved with respect to the unknowns β and γ analytically. Therefore it is necessary to solve it by means of the numerical methods. For the following data

$$l_0 = 0.1m, \qquad l_1 = 0.02m, \qquad l_2 = 0.09m, \qquad l_3 = 0.04m \tag{3.51}$$

the numerical solution of the equations 3.50 is presented in Fig. 25a). The result of the numerical differentiation of these functions with respect to the angular displacement α is shown in Fig. 25b) and c). These diagrams allows the angular velocity $\dot{\beta}$ and $\dot{\gamma}$



3
2.5
2
1.5
1
0.5
0
[rad]
γ
β
a)
0
1
2
3
4
5
6
7
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
$\frac{d\gamma}{d\alpha}$
$\frac{d\beta}{d\alpha}$
b)
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
$\frac{d^2\beta}{d\alpha^2}$
$\frac{d^2\gamma}{d\alpha^2}$
c)
α [rad]

Figure 25

as well as the angular accelerations $\ddot{\beta}$ and $\ddot{\gamma}$ to be computed.

$$\begin{aligned}
\dot{\beta} &= \frac{d\beta}{d\alpha}\frac{d\alpha}{dt} = \frac{d\beta}{d\alpha}\dot{\alpha} \\
\dot{\gamma} &= \frac{d\gamma}{d\alpha}\frac{d\alpha}{dt} = \frac{d\gamma}{d\alpha}\dot{\alpha} \\
\ddot{\beta} &= \frac{d}{dt}\left(\frac{d\beta}{d\alpha}\dot{\alpha}\right) = \frac{d^2\beta}{d\alpha^2}\frac{d\alpha}{dt}\dot{\alpha} + \frac{d\beta}{d\alpha}\ddot{\alpha} = \frac{d^2\beta}{d\alpha^2}\dot{\alpha}^2 + \frac{d\beta}{d\alpha}\ddot{\alpha} \\
\ddot{\gamma} &= \frac{d}{dt}\left(\frac{d\gamma}{d\alpha}\dot{\alpha}\right) = \frac{d^2\gamma}{d\alpha^2}\frac{d\alpha}{dt}\dot{\alpha} + \frac{d\gamma}{d\alpha}\ddot{\alpha} = \frac{d^2\gamma}{d\alpha^2}\dot{\alpha}^2 + \frac{d\gamma}{d\alpha}\ddot{\alpha}
\end{aligned} \tag{3.52}$$

Fig. 26 displays the free body diagram of the link3. Since the link 2 is massless, the line of action of the reaction $\mathbf{R}_{32}$ must coincide the line BC. The Newton's generalized equation applied to the link 3 yields

$$I_D\ddot{\gamma} = -m_3gc_3\cos\gamma - R_{32}l_3\sin(\gamma-\beta) \tag{3.53}$$

For the link 1 (see Fig. 27) we can write

$$0 = M + R_{32}l_1\sin(\alpha-\beta) \tag{3.54}$$

Hence

$$R_{32} = -\frac{M}{l_1\sin(\alpha-\beta)} \tag{3.55}$$

Introduction of 3.55 into the equation 3.53 results in the following equation with the only one unknown α.

$$I_D\ddot{\gamma} + m_3gc_3\cos\gamma - \frac{M}{l_1\sin(\alpha-\beta)}l_3\sin(\gamma-\beta) = 0 \tag{3.56}$$

Since according to 3.52

$$\ddot{\gamma} = \frac{d^2\gamma}{d\alpha^2}\dot{\alpha}^2 + \frac{d\gamma}{d\alpha}\ddot{\alpha} \tag{3.57}$$

The equation of motion takes form

$$I_D\frac{d\gamma}{d\alpha}\ddot{\alpha} + I_D\frac{d^2\gamma}{d\alpha^2}\dot{\alpha}^2 + m_3gc_3\cos\gamma(\alpha) - \frac{M}{l_1\sin(\alpha-\beta(\alpha))}l_3\sin(\gamma(\alpha)-\beta(\alpha)) = 0 \tag{3.58}$$

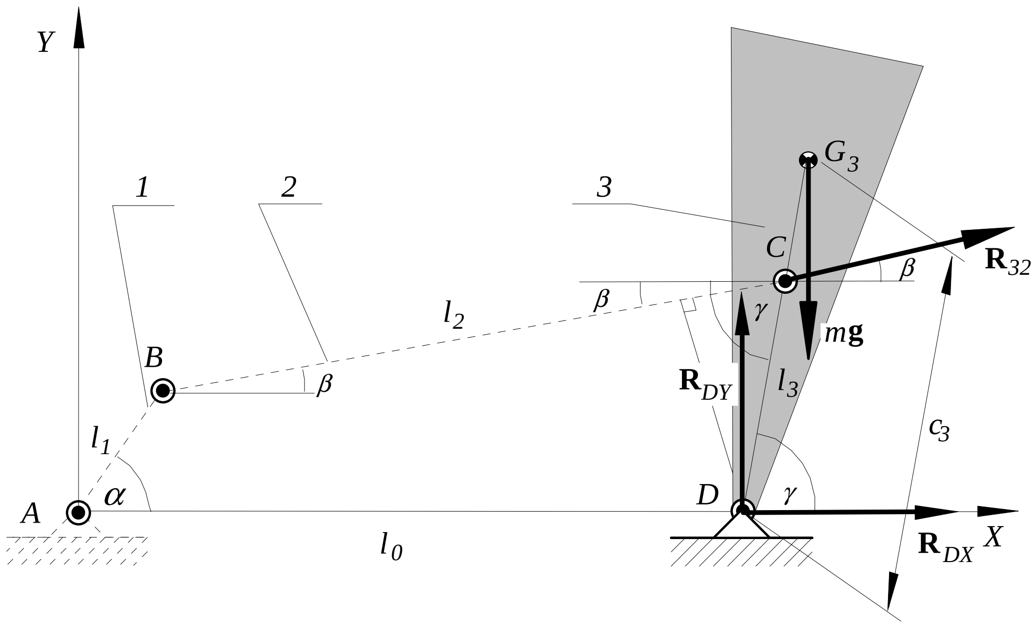


Figure 26

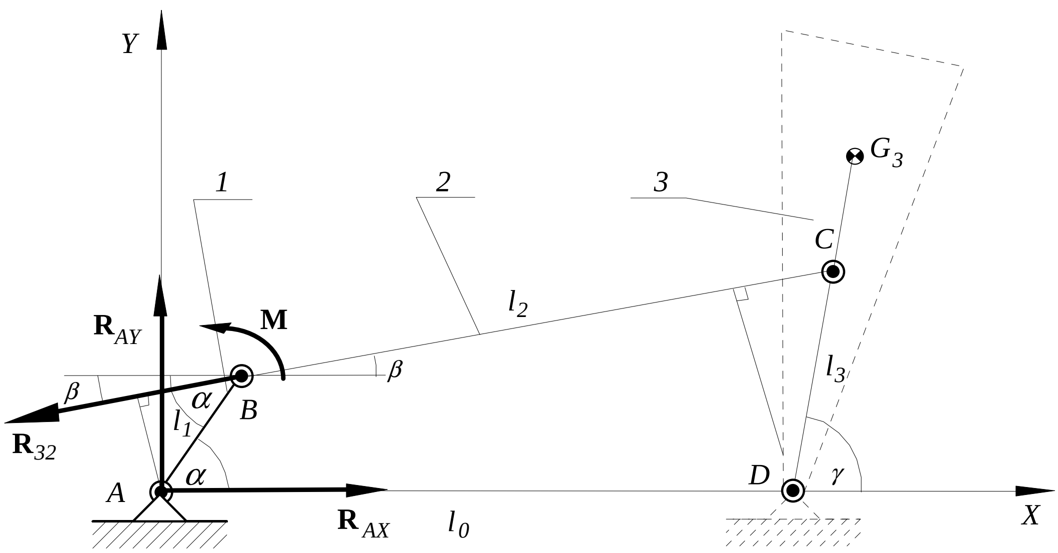


Figure 27

Problem 50

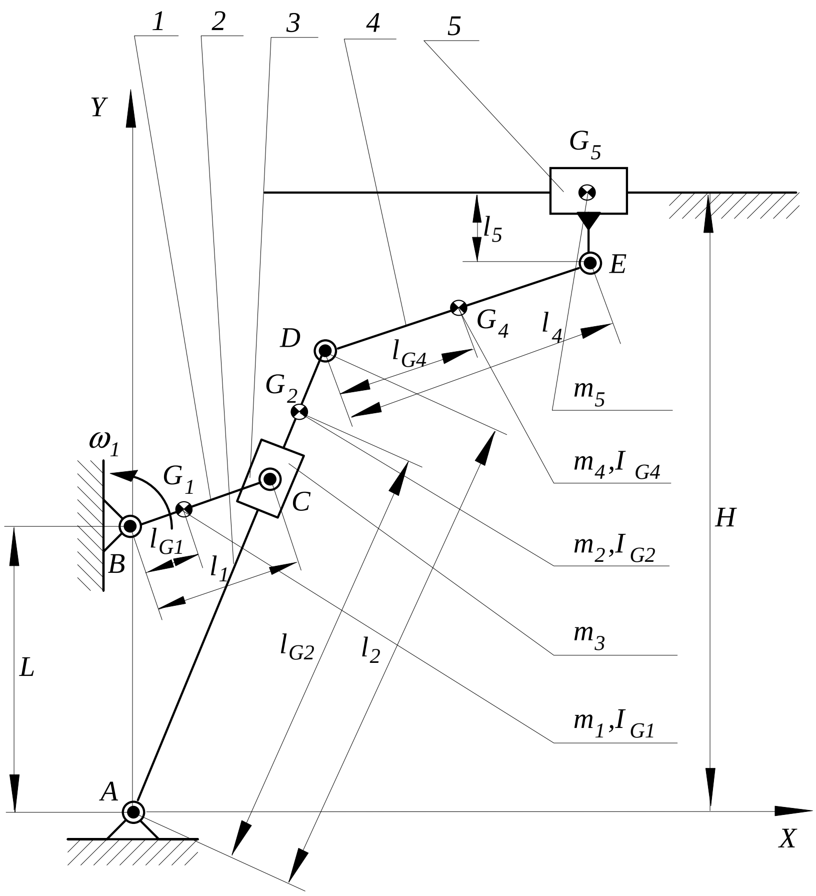


Figure 28

The link 1 of the mechanism presented in Fig. 28 rotates with the constant angular velocity ω_1 in the direction shown. The points denoted by G represents position of the centre of gravity of the individual links. The symbols m and I refers to the mass and the moment of inertia of the individual links. Produce equations that determine the interaction forces in all joints and the driving moment that must be applied to the driving link 1 to assure its motion with the angular velocity ω_1.

Problem 51

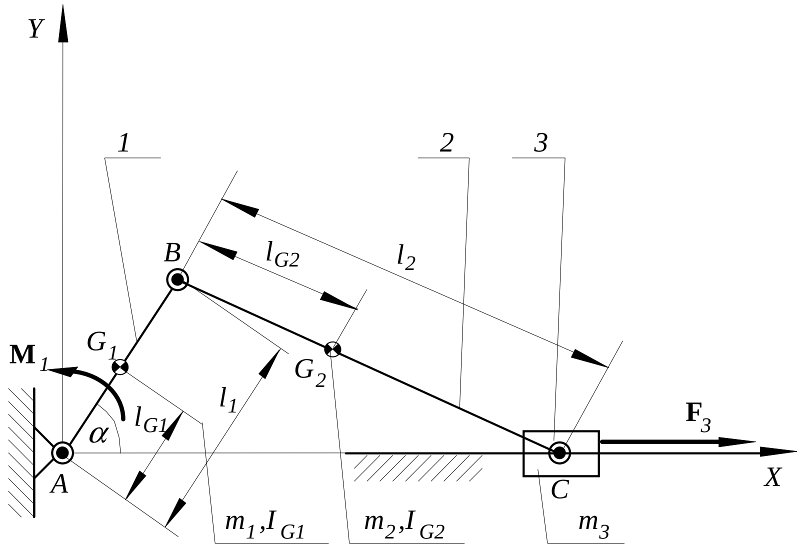


Figure 29

The compressor presented in Fig. 29 is driven by a motor. Its driving moment $M_1(\dot{\alpha})$ is a known function of the angular velocity $\dot{\alpha}$ of the link 1. $F_3(\alpha)$represents the output force that is a known function of the angular position $\alpha$of the link 1. The points denoted by G represents position of the centre of gravity of the individual links. The symbols m and I refers to the mass and the moment of inertia of the individual links. Produce the differential equation of motion of the mechanism

Solution
KINEMATIC ANALYSIS

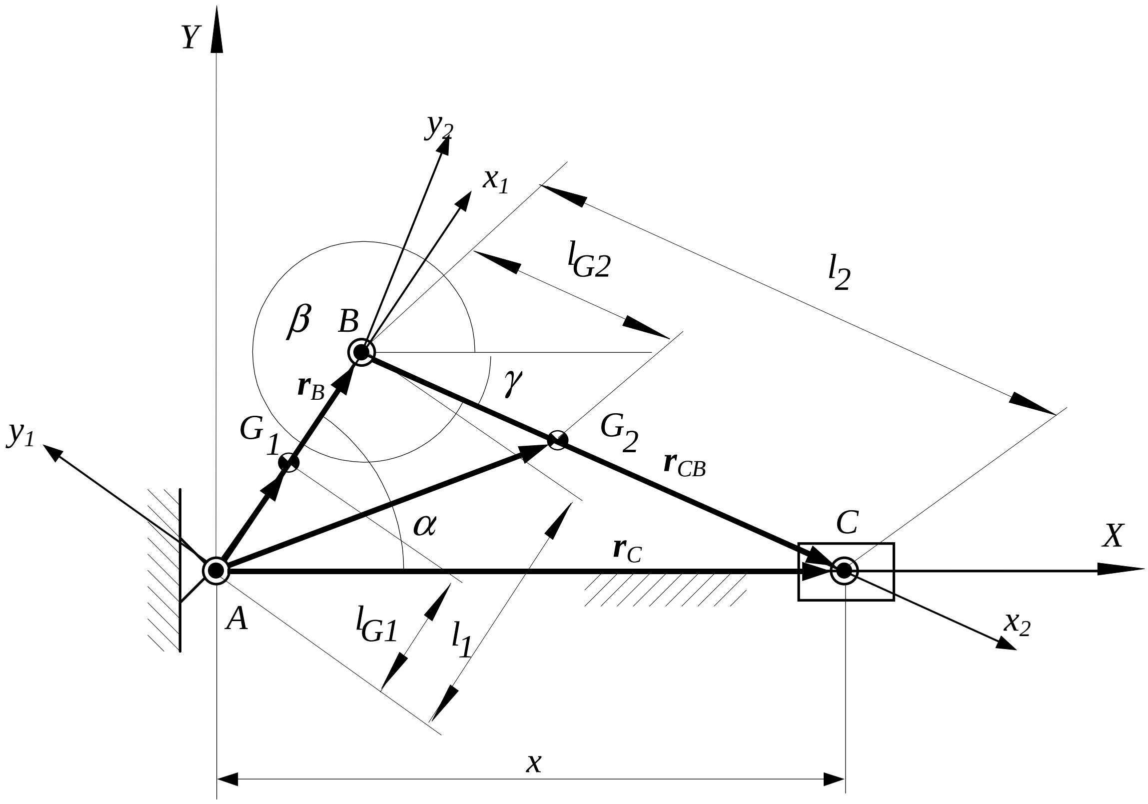


Figure 30

To determine the unknow parameters x and γ as a function of time let us consider the following vector equation (see Fig. 30)

$$\mathbf{r}_B + \mathbf{r}_{CB} = \mathbf{r}_C \tag{3.59}$$

It can be rewritten

$$\mathbf{i}_1 l_1 + \mathbf{i}_2 l_2 = \mathbf{I} x \tag{3.60}$$

Multiplication of the above equation by the unit vectors $\mathbf{I}$ and $\mathbf{J}$ yields set of two scalar equations with the two unknows x and γ

$$\begin{aligned} l_1 \cos\alpha + l_2 \cos\gamma &= x \\ l_1 \sin\alpha - l_2 \sin\gamma &= 0 \end{aligned} \tag{3.61}$$

Its solution is

$$\begin{aligned} \gamma &= \arcsin\left(\frac{l_1}{l_2}\sin\alpha\right) \\ x &= l_1 \cos\alpha + l_2 \cos\gamma \end{aligned} \tag{3.62}$$

The kinematic parameters of the individual links

Link 1

angular velocity

$$\boldsymbol{\omega}_1 = \mathbf{K}\dot{\alpha} \tag{3.63}$$

angular acceleration

$$\boldsymbol{\varepsilon}_1 = \mathbf{K}\ddot{\alpha} \tag{3.64}$$

linear velocity of centre of gravity G_1

$$\mathbf{r}_{G1} = \mathbf{i}_1 l_{G1} \tag{3.65}$$

$$\mathbf{v}_{G1} = \dot{\mathbf{r}}_{G1} = \mathbf{r}'_{G1} + \boldsymbol{\omega}_1 \times \mathbf{r}_{G1} = \begin{vmatrix} \mathbf{i}_1 & \mathbf{j}_1 & \mathbf{k}_1 \\ 0 & 0 & \dot{\alpha} \\ l_{G1} & 0 & 0 \end{vmatrix} = \mathbf{j}_1 l_{G1}\dot{\alpha} \tag{3.66}$$

linear acceleration of centre of gravity G_1

$$\mathbf{a}_{G1} = \dot{\mathbf{v}}_{G1} = \mathbf{v}'_{G1} + \boldsymbol{\omega}_1 \times \mathbf{v}_{G1} = \mathbf{j}_1 l_{G1}\ddot{\alpha} + \begin{vmatrix} \mathbf{i}_1 & \mathbf{j}_1 & \mathbf{k}_1 \\ 0 & 0 & \dot{\alpha} \\ 0 & l_{G1}\dot{\alpha} & 0 \end{vmatrix} = \mathbf{j}_1 l_{G1}\ddot{\alpha} + \mathbf{i}_1 l_{G1}\dot{\alpha}^2 \tag{3.67}$$

Link 2

angular velocity

$$\boldsymbol{\omega}_2 = \mathbf{K}\dot{\beta} = \mathbf{K}\frac{d}{dt}(360^o - \gamma) = \mathbf{K}(-\dot{\gamma}) = \mathbf{K}\left(-\frac{d}{dt}\arcsin\left(\frac{l_1}{l_2}\sin\alpha\right)\right) \tag{3.68}$$

angular acceleration

$$\boldsymbol{\varepsilon}_2 = \mathbf{K}(-\ddot{\gamma}) \tag{3.69}$$

linear velocity of centre of gravity G_1

$$\begin{aligned} \mathbf{r}_{G2} &= \mathbf{i}_1 l_1 + \mathbf{i}_2 l_{G2} = (\mathbf{i}_2 \sin(90 - \alpha - \gamma) + \mathbf{j}_2 \cos(90 - \alpha - \gamma)\, l_1 + \mathbf{i}_2 l_{G2} = \\ &= \mathbf{i}_2 \left(l_1 \cos(\alpha + \gamma) + l_{G2}\right) + \mathbf{j}_2 l_1 \sin(\alpha + \gamma) \end{aligned} \tag{3.70}$$

$$\begin{aligned} \mathbf{v}_{G2} &= \dot{\mathbf{r}}_{G2} = \mathbf{r}'_{G2} + \boldsymbol{\omega}_2 \times \mathbf{r}_{G2} \\ &= \mathbf{i}_2 \left(-l_1(\dot{\alpha} + \dot{\gamma}) \sin(\alpha + \gamma)\right) + \mathbf{j}_2 \left(l_1(\dot{\alpha} + \dot{\gamma}) \cos(\alpha + \gamma)\right) \\ &\qquad + \begin{vmatrix} \mathbf{i}_2 & \mathbf{j}_2 & \mathbf{k}_2 \\ 0 & 0 & -\dot{\gamma} \\ l_1 \cos(\alpha + \gamma) + l_{G2} & l_1 \sin(\alpha + \gamma) & 0 \end{vmatrix} \\ &= \mathbf{i}_2 \left(v_{G2x2}\right) + \mathbf{j}_2 \left(v_{G2y2}\right) \end{aligned} \tag{3.71}$$

linear acceleration of centre of gravity G_1

$$\begin{aligned} &\mathbf{a}_{G2} = \dot{\mathbf{v}}_{G2} = \mathbf{v}'_{G2} + \boldsymbol{\omega}_2 \times \mathbf{v}_{G2} = \\ &= \mathbf{i}_2 \left(\dot{v}_{G2x2}\right) + \mathbf{j}_2 \left(\dot{v}_{G2y2}\right) + \begin{vmatrix} \mathbf{i}_2 & \mathbf{j}_2 & \mathbf{k}_2 \\ 0 & 0 & -\dot{\gamma} \\ v_{G2x2} & v_{G2y2} & 0 \end{vmatrix} = \\ &= \mathbf{i}_2 \left(\dot{v}_{G2x2} + \dot{\gamma} v_{G2y2}\right) + +\mathbf{j}_2 \left(\dot{v}_{G2y2} - \dot{\gamma} v_{G2x2}\right) \end{aligned} \tag{3.72}$$

Link 3
angular velocity

$$\boldsymbol{\omega}_3 = \mathbf{K}0 \tag{3.73}$$

angular acceleration

$$\boldsymbol{\varepsilon}_3 = \mathbf{K}0 \tag{3.74}$$

linear velocity of centre of gravity C

$$\mathbf{r}_C = \mathbf{I}x = \mathbf{I}\left(l_1 \cos\alpha + l_2 \cos\gamma\right) \tag{3.75}$$

$$\mathbf{v}_C = \dot{\mathbf{r}}_C = \mathbf{I}\left(-l_1\dot{\alpha}\sin\alpha + l_2\dot{\gamma}\sin\gamma\right) \tag{3.76}$$

linear acceleration of centre of gravity C

$$\mathbf{a}_C = \dot{\mathbf{v}}_C = \mathbf{I}\left(-l_1\ddot{\alpha}\sin\alpha - l_1\dot{\alpha}^2\cos\alpha + l_2\ddot{\gamma}\sin\gamma + l_2\dot{\gamma}^2\cos\gamma\right) \tag{3.77}$$

KINETIC ANALYSIS

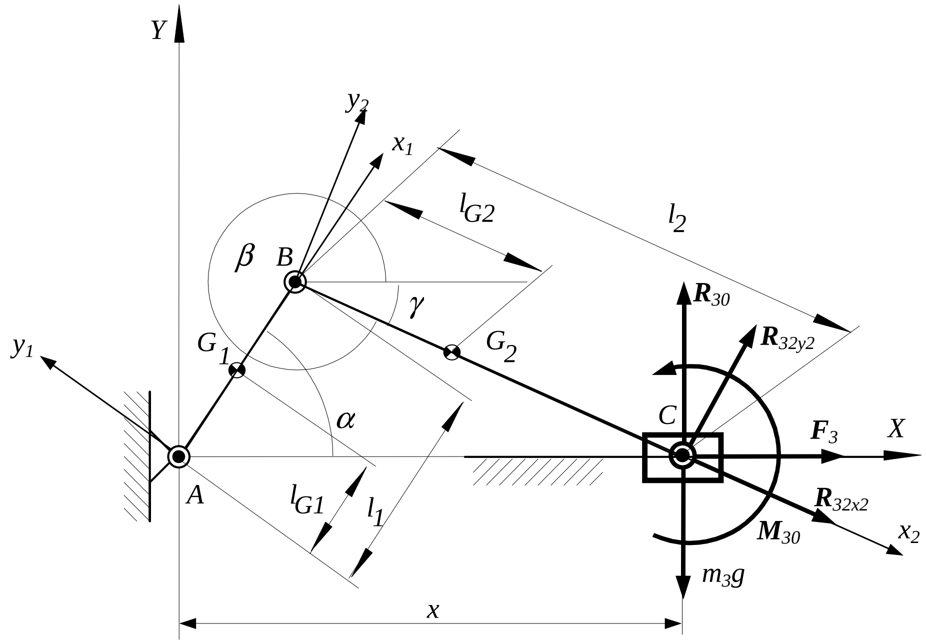


Figure 31

Link 3
According to the free body daigram of the link1, shown in Fig. 31, one can produce the following equations

$$m_3\ddot{x} = +F_3 + R_{32x2}\cos\gamma + R_{32y2}\sin\gamma \tag{3.78}$$
$$0 = -m_3 g + R_{30} - R_{32x2}\sin\gamma + R_{32y2}\cos\gamma \tag{3.79}$$
$$0 = M_{03} \tag{3.80}$$

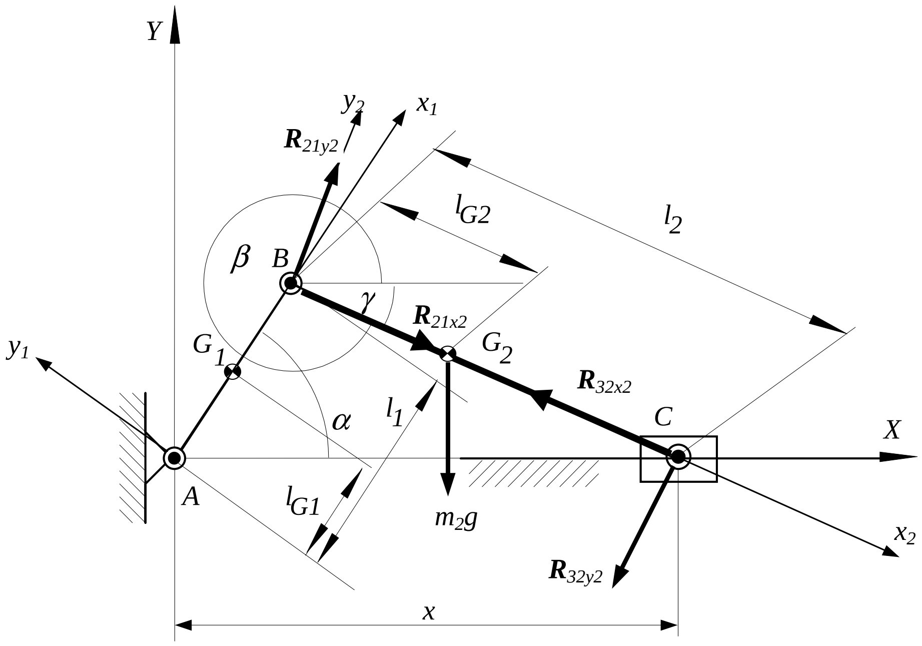


Figure 32

Link 2

According to the shown in Fig. 32 free body daigram of the link2 one can produce the following equations

$$m_2 a_{G2x2} = +m_2 g \sin\gamma - R_{32x2} + R_{21x2} \tag{3.81}$$

$$m_2 a_{G2y2} = -m_2 g \cos\gamma - R_{32y2} + R_{21y2} \tag{3.82}$$

$$I_{G2}\varepsilon_2 = -R_{32y2}(l_2 - l_{G2}) - R_{21y2} l_{G2} \tag{3.83}$$

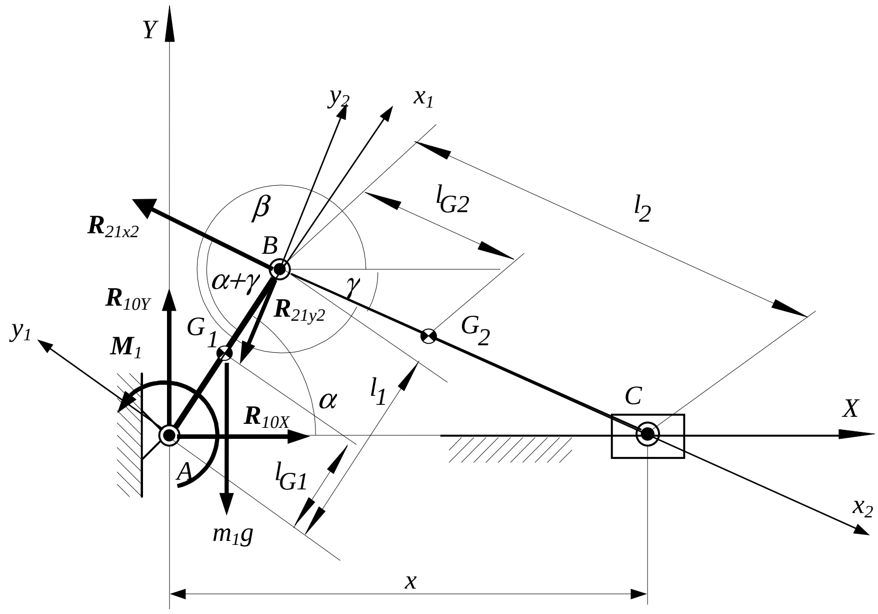


Figure 33

Link 1

According to the shown in Fig. 33 free body daigram of the link1 one can produce the following equations

$$
\begin{aligned}
m_1 a_{G1x1} &= -m_1 g \sin\alpha - R_{21x2}\cos(\alpha+\gamma) - R_{21y2}\sin(\alpha+\gamma) + R_{10X}\cos\alpha + R_{10Y}\sin\alpha \\
m_1 a_{G2y1} &= -m_1 g \sin\alpha + R_{21x2}\sin(\alpha+\gamma) - R_{21y2}\cos(\alpha+\gamma) - R_{10X}\cos\alpha + R_{10Y}\sin\alpha \\
I_{1A}\varepsilon_1 &= M_1 - m_1 g l_{G1}\cos\alpha + R_{21x2} l_1 \cos(\alpha+\gamma) - R_{21y2} l_1 \sin(\alpha+\gamma) \qquad (3.84)
\end{aligned}
$$

The equations 3.78, 3.81, 3.80 and 3.81

$$
\begin{bmatrix} 0 & 0 & \cos\gamma & \sin\gamma \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -(l_2 - l_{G2}) & 0 & -l_{G2} \end{bmatrix}
\begin{bmatrix} R_{21x2} \\ R_{21y2} \\ R_{32x2} \\ R_{32y2} \end{bmatrix}
=
\begin{bmatrix} +m_3\ddot{x} - F_3 \\ m_2 a_{G2x2} - m_2 g \sin\gamma \\ m_2 a_{G2y2} + m_2 g \cos\gamma \\ I_{G2}\varepsilon_2 \end{bmatrix} \qquad (3.85)
$$

can be solved for the unknown interaction forces R_{21x2}, R_{21y2}, R_{32x2}, and R_{32y2}

$$
\begin{aligned}
\begin{bmatrix} R_{21x2} \\ R_{21y2} \\ R_{32x2} \\ R_{32y2} \end{bmatrix}
&=
\begin{bmatrix} 0 & 0 & \cos\gamma & \sin\gamma \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -(l_2 - l_{G2}) & 0 & -l_{G2} \end{bmatrix}^{-1}
\begin{bmatrix} +m_3\ddot{x} - F_3 \\ m_2 a_{G2x2} - m_2 g \sin\gamma \\ m_2 a_{G2y2} + m_2 g \cos\gamma \\ I_{G2}\varepsilon_2 \end{bmatrix} \\
&= \begin{bmatrix} [A_{21}] \\ [A_{32}] \end{bmatrix}
\begin{bmatrix} +m_3\ddot{x} - F_3 \\ m_2 a_{G2x2} - m_2 g \sin\gamma \\ m_2 a_{G2y2} + m_2 g \cos\gamma \\ I_{G2}\varepsilon_2 \end{bmatrix} \qquad (3.86)
\end{aligned}
$$

Hence

$$\begin{bmatrix} R_{21x2} \\ R_{21y2} \end{bmatrix} = [A_{21}] \begin{bmatrix} +m_3\ddot{x} - F_3 \\ m_2 a_{G2x2} - m_2 g \sin\gamma \\ m_2 a_{G2y2} + m_2 g \cos\gamma \\ I_{G2}\varepsilon_2 \end{bmatrix} \tag{3.87}$$

On the other hand the equation 3.84 can be rewritten as following

$$I_{1A}\varepsilon_1 - M_1 + m_1 g l_{G1} \cos\alpha + \begin{bmatrix} -l_1\cos(\alpha+\gamma) & +l_1\sin(\alpha+\gamma) \end{bmatrix} \begin{bmatrix} R_{21x2} \\ R_{21y2} \end{bmatrix} \tag{3.88}$$

Hence, taking into account 3.87 the equation of motion of the system is

$$\begin{aligned} & I_{1A}\varepsilon_1 - M_1 + m_1 g l_{G1} \cos\alpha + \\ & + \begin{bmatrix} -l_1\cos(\alpha+\gamma) & +l_1\sin(\alpha+\gamma) \end{bmatrix} \left([A_{21}] \begin{bmatrix} +m_3\ddot{x} - F_3 \\ m_2 a_{G2x2} - m_2 g \sin\gamma \\ m_2 a_{G2y2} + m_2 g \cos\gamma \\ I_{G2}\varepsilon_2 \end{bmatrix} \right) = 0 \end{aligned} \tag{3.89}$$

Problem 52

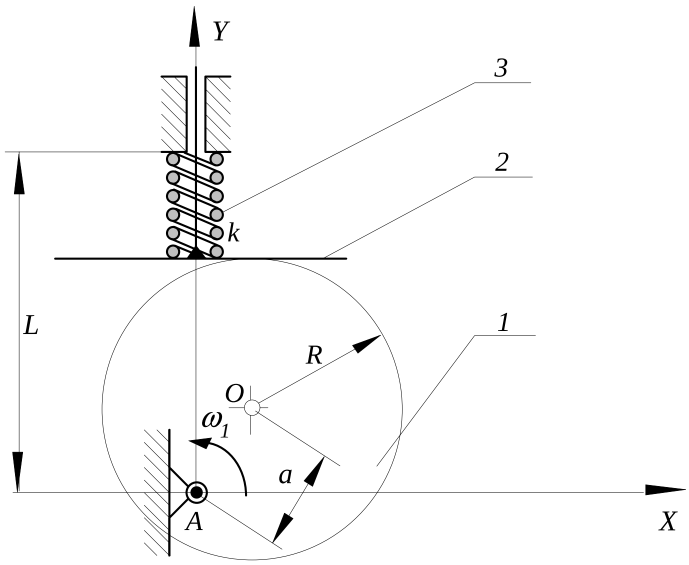


Figure 34

The cam 1 shown in Fig. 34 rotates with the constant velocity ω_1. The follower 2 of mass m is kept in contact with the cam due to gravitation and the spring 3. The uncompressed length of the spring is l and its stiffness is k. The friction between the cam and the follower is negligible. Produce expression for the interaction force between the cam and the follower.

Answer:

$N = mg - ma\omega_1^2 \sin\omega_1 t + k(l - L + Ra \sin\omega_1 t$

Problem 53

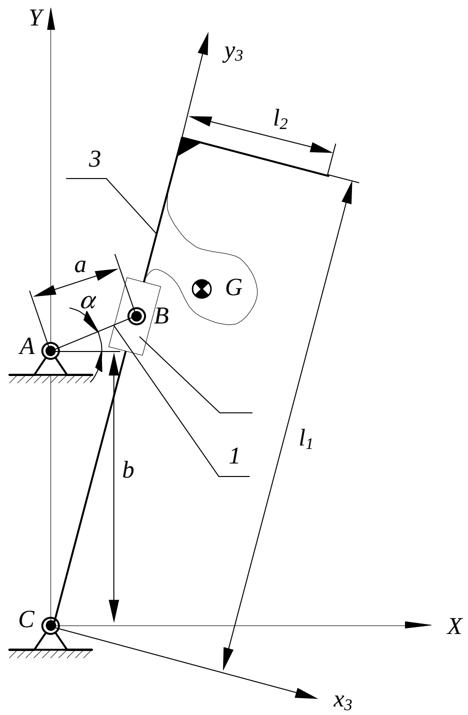


Figure 35

The mechanism shown in Fig. 35 moves in the vertical plane XY of the inertial system of coordinates XYZ. It is driven by the link 1. The angular position of the link 1 as a function of time is determined by the angular displacement α. Two uniform and slender rods of length l1 and l2 and the same unit mass ρ were joined together to form the link 3. The system of coordinates x_3y_3 is rigidly attached to the link 3. The links 1 and 2 are massless.

Produce

1. the mobility of the mechanism

Answer:

$MO = 1$

2. the expression for the absolute angular velocity of the link 3

Answer:

$\omega_3 = \frac{d}{dt}\left[\arcsin\frac{b+a\sin\alpha}{\sqrt{a^2+b^2+2ab\sin\alpha}}\right]$

3. the expression for the absolute angular acceleration of the link 3

Answer:

$\varepsilon_3 = \frac{d}{dt}\left[\frac{d}{dt}\left[\arcsin\frac{b+a\sin\alpha}{\sqrt{a^2+b^2+2ab\sin\alpha}}\right]\right]$

4. the expression for the moment of inertia of the link 3 about the centre of gravity G

5. the expression for the moment of inertia of the link 3 about the cntre of rotation C.

6. the expression for the trajectory of the centre of gravity G of the link 3
7. the expression for the components of the absolute velocity of the point G along the system of coordinates $x_3 y_3$
8. the expression for the kinetic energy of the link 3
9. the components of the reactions at the constraints C
10. the expression for the driving moment that has to be applied to the link 1 to maintain it moving according to the assumed motion α.

Problem 54

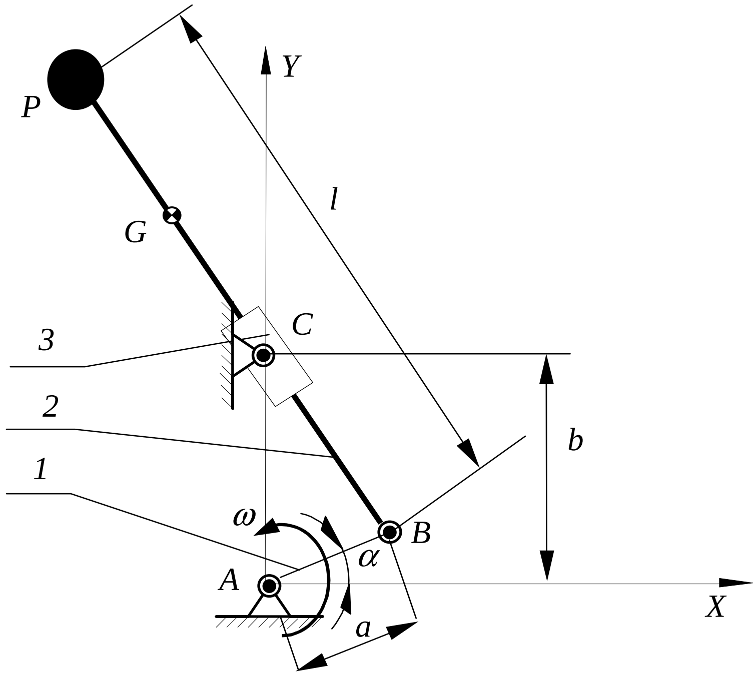


Figure 36

The mechanism shown in Fig. 36moves in the vertical plane XY of the inertial system of coordinates XYZ. It is driven by the massless link 1. The link 1 rotates with the constant angular velocity ω. The uniform and slender rod of length l and mass M and the particle P of mass m are joined together to form the link 2. The link 3 is massless.

Produce

1. the mobility of the mechanism
2. the expression for the absolute angular velocity of the link 2 and 3
3. the expression for the absolute angular acceleration of the link 3
4. the expression for the trajectory of the centre of gravity G of the link 2
5. the expression for the components of the absolute velocity of the point G
6. the expression for the moment of inertia of the link 2 about its centre of gravity G
7. the expression for the kinetic energy of the link 2
8. the components of the reactions at the constraints A, B and C
9. the expression for the driving moment that has to be applied to the link 1 to maintain its motion with the constant velocity ω.

Problem 55

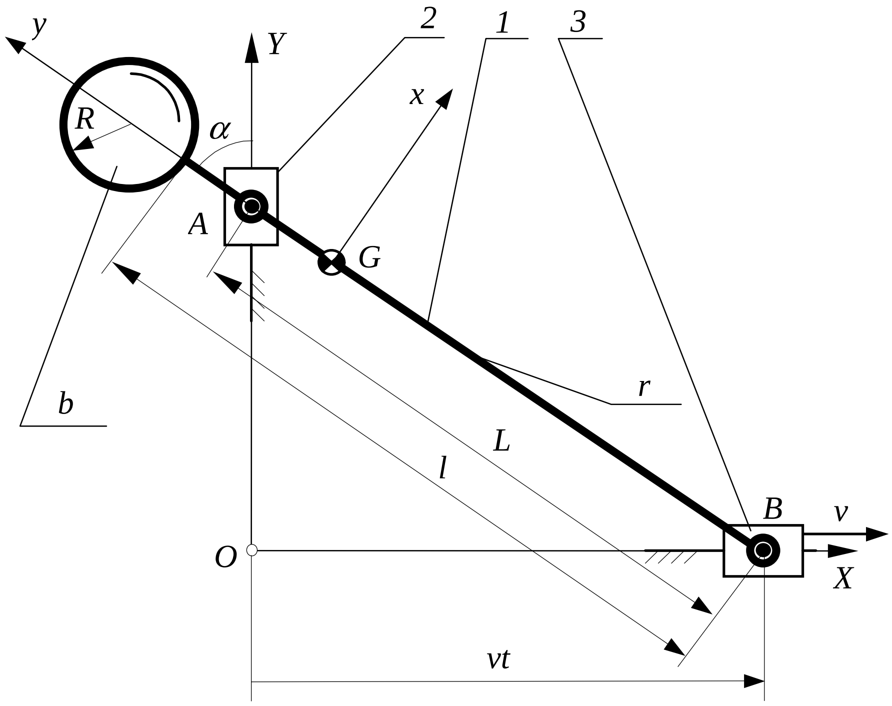


Figure 37

The link 1 of the mechanism shown in Fig. 37, which can be treated as a rigid body, is assembled of the ball b and a uniform rod r. The mass of the ball and its radius are m_b and R respectively whereas m_r and l stand for mass and length of the rod respectively. The sliders 2 and 3 are massless and can move along axes X and Y of the inertial system of coordinates XYZ. The slider 3 has a constant velocity v.

Produce:

1. the expression for the position of the centre of gravity G

Answer:

$BG = \frac{m_r \frac{l}{2} + m_b(l+R)}{m_r + m_b}$

2. the expression for the moment of inertia of the link 1 about the axis through its centre of gravity G

Answer:

$I_G = \frac{2}{5} m_b R^2 + m_b(L + R - BG)^2 + \frac{1}{12} m_r l^2 + m_r(BG - \frac{l}{2})^2$

3. the expression for the absolute angular velocity of the link 1

Answer:

$\omega_1 = \frac{v}{L} \frac{1}{\sqrt{1 - \left(\frac{vt}{L}\right)^2}}$

4. the expression for the components of the absolute velocity of the centre of gravity G of the link 1along the inertial system of coordinates XYZ

Answer:

$\mathbf{v}_G = \mathbf{I}(v - BG\dot{\alpha}\cos\alpha) + \mathbf{J}(-BG\dot{\alpha}\sin\alpha)$

where: $\alpha = \arcsin\frac{vt}{L}$

5. the expression for the components of the absolute acceleration of the centre of gravity G of the link 1along the inertial system of coordinates XYZ

Answer:

$\mathbf{a}_G = \mathbf{I}(-BG\ddot{\alpha}\cos\alpha + BG\dot{\alpha}^2\sin\alpha) + \mathbf{J}(-BG\ddot{\alpha}\sin\alpha - BG\dot{\alpha}^2\cos\alpha)$

6. the expression for the kinetic energy of the link 1

Answer:

$T = \frac{1}{2}m\mathbf{v}_G^2 + \frac{1}{2}I_G\omega_1^2$

Problem 56

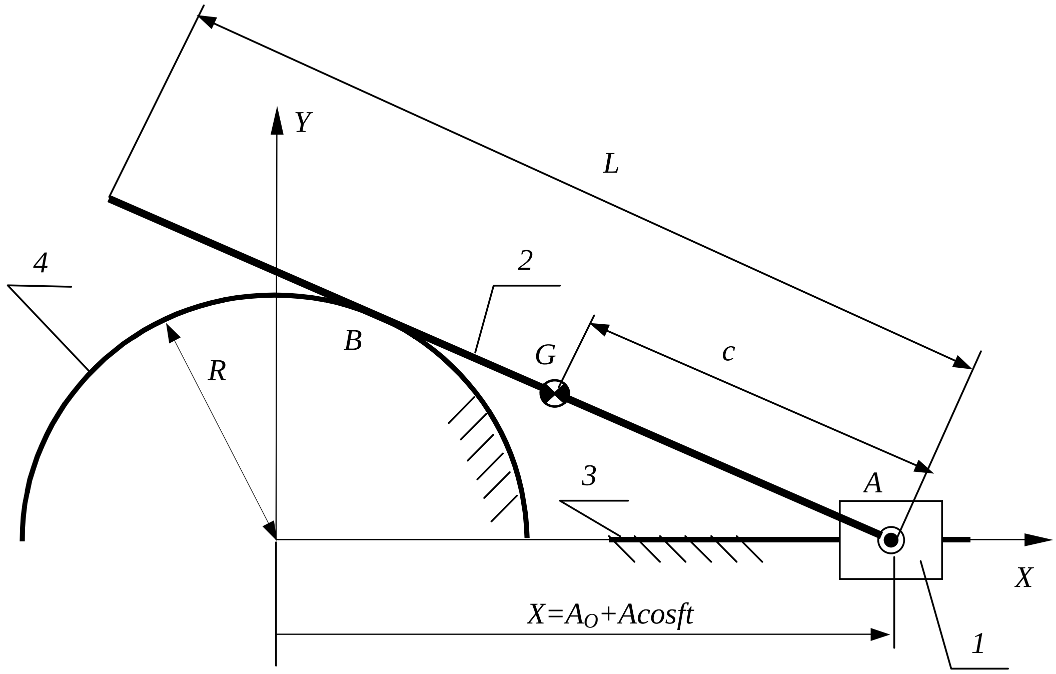


Figure 38

The link 1 of the mechanism shown in Fig 38 moves along the slide 3 that coincides with the horizontal axis X of the inertial system of coordinates XYZ. Its motion is defined by the following function of time.

$$X = A_O + Acosft$$

where f stands for the frequency and A stands for the amplitude of this harmonic motion.
The link 2 is hinged to the link 1 at the point A and is supported by the cylindrical surface 4 at the point B. The cylindrical surface is of radius R. The link 2 of length L possesses mass m and its moment of inertia about the axis through the center of gravity G is I_G. The distance c locates the position of the center of gravity. Friction in all the kinematic constraints may be neglected.

Produce:

1. The mobility and number of degrees of freedom of the mechanism.

Answer:

$MO = 1, \qquad M = 0$

2. The expression for the absolute angular velocity of the link 2.

Answer:

$\omega_2 = \frac{-RAf \sin ft}{\sqrt{(A_O+A\cos ft)^2-R^2}(A_O+A\cos ft)}$

3. The expression for the absolute velocity of the centre of gravity G of the link 2

Answer:

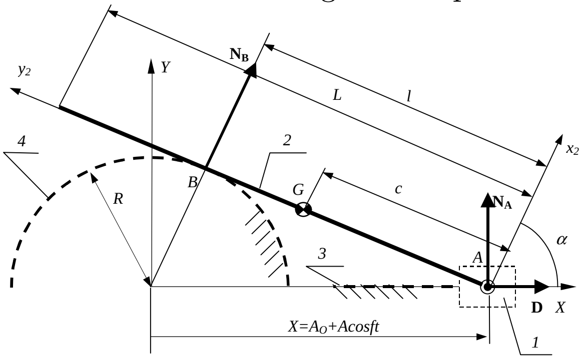
$\mathbf{v}_G = \mathbf{I}(\dot{X} - c\dot{\alpha}\cos\alpha) + \mathbf{J}(-c\dot{\alpha}\sin\alpha)$

4 The expressions for the interaction forces that act on the link 2 at the points A and B

5. The expression for the driving force that must be applied to the link 1 to maintain the assumed motion.

Answer to question 4 and 5:

Solution of the following set of equations with respect to N_A, N_B and D respectively



$$ma_X = D + N_B\cos\alpha$$
$$ma_Y = -mg + N_B\sin\alpha + N_A$$
$$I_G\ddot{\alpha} = -N_B(l-c) + Dc\cos\alpha + N_A c\sin\alpha$$

where:

$$a_X = \ddot{X} - c\ddot{\alpha}\cos\alpha + c\dot{\alpha}^2\sin\alpha$$
$$a_Y = -c\ddot{\alpha}\sin\alpha - c\dot{\alpha}^2\cos\alpha$$

Problem 57

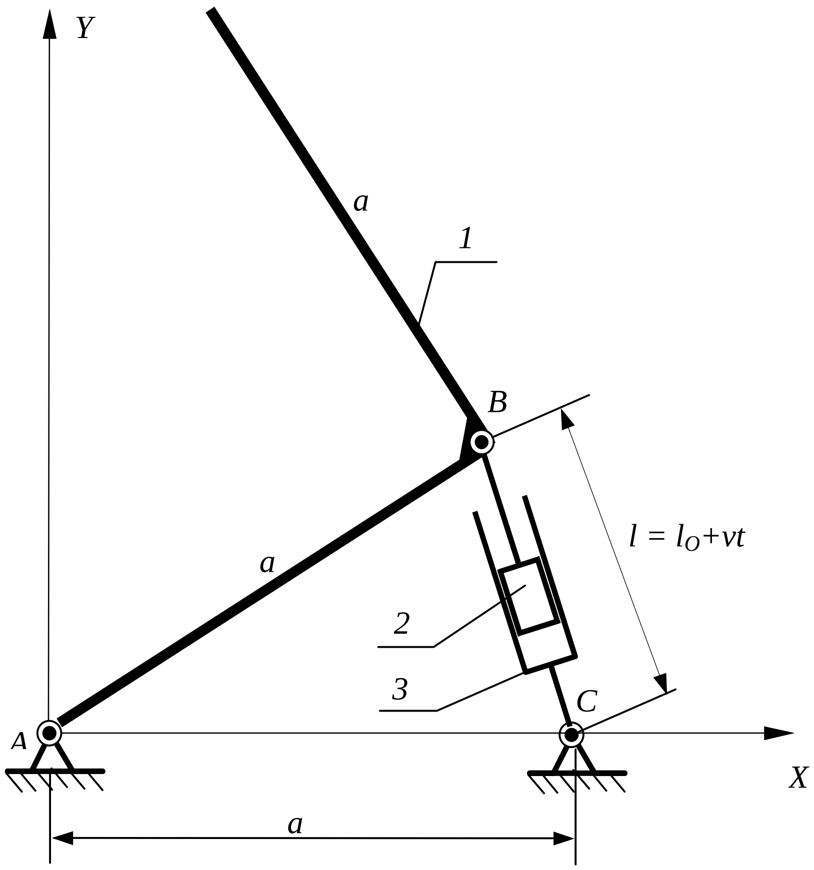


Figure 39

The mechanism shown in Fig. 39 operates in the vertical plane of the inertial space XYZ. Two uniform rods of mass m and length a are joined together to form the link *1* of this mechanism. This link is hinged at the point A to the inertial space and at the point B is supported by the actuator *2-3*. The piston *2* and the cylinder *3* of the actuator are massless. The length l of the actuator varies in time according to the following function

$$l = l_o + \mathrm{v}t$$

Produce:

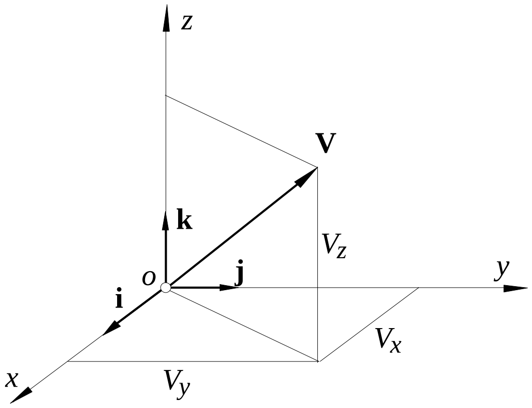
1. The mobility of the mechanism and its number of degrees of freedom
2. The expression for the moment of inertia of the link *1* about the axis through the point of rotation A
3. The expression for the absolute angular velocity of the link *1*
4. The expression for the absolute velocity of the center of gravity of the link *1*
5. The expression for the kinetic energy stored in the link *1*
6. The free body diagrams for the link *1* and the actuator *2-3*

Chapter 4

APPENDIXES

4.1 APPENDIX 1. REVISION OF THE VECTOR CALCULUS

NOTATION.



The **vector quantities** are printed in boldface type **V** or, in handwriting, should always be indicated by symbol $\underline{V}$ to distinguish them from the scalar quantities V. Vector quantities are usually defined in the **right-handed** system of coordinates by its scalar components V_x, V_y, V_z.

$$\mathbf{V} = \mathbf{i}V_x + \mathbf{j}V_y + \mathbf{k}V_z \tag{4.1}$$

were $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are **unit vectors** of the system of coordinates xyz.

SCALAR MAGNITUDE of a vector **V** is

$$V = \sqrt{V_x^2 + V_y^2 + V_z^2} \tag{4.2}$$

DIRECTION COSINES l, m, n are the cosines of angles between a vector **V** and axes xyz.. Thus

$$l = \cos\measuredangle\mathbf{Vi} = \frac{V_x}{V} \qquad n = \cos\measuredangle\mathbf{Vj} = \frac{V_y}{V} \qquad m = \cos\measuredangle\mathbf{Vk} = \frac{V_z}{V} \tag{4.3}$$

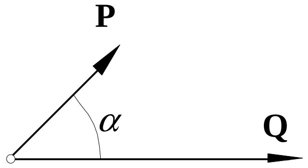
Useful relations

$$l^2 + n^2 + m^2 = 1 \tag{4.4}$$

DOT OR SCALAR PRODUCT of two vectors **P** and **Q** is defined as scalar magnitude

$$\mathbf{P} \cdot \mathbf{Q} = PQ\cos\alpha \tag{4.5}$$

where α stands for the angle between the two vectors.



Useful relations

$$\mathbf{P} \cdot \mathbf{Q} = \mathbf{Q} \cdot \mathbf{P} \tag{4.6}$$

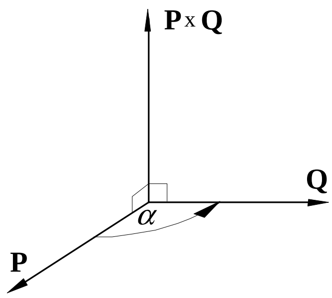
$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = \mathbf{1} \tag{4.7}$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{i} \cdot \mathbf{j} = \mathbf{0} \tag{4.8}$$

$$\mathbf{P} \cdot \mathbf{Q} = \begin{bmatrix} P_x & P_y & P_z \end{bmatrix} \begin{bmatrix} Q_x \\ Q_y \\ Q_z \end{bmatrix} = P_xQ_x + P_yQ_y + P_zQ_z \tag{4.9}$$

$$P_x = \mathbf{i} \cdot \mathbf{P} \qquad P_y = \mathbf{j} \cdot \mathbf{P} \qquad P_z = \mathbf{k} \cdot \mathbf{P} \tag{4.10}$$

VECTOR OR CROSS PRODUCT of two vectors **P** and **Q** is defined as a vector with the magnitude $PQ\sin\alpha$ and direction specified by the right-hand rule as shown.



Useful relations

$$\mathbf{P} \times (\mathbf{Q} + \mathbf{R}) = \mathbf{P} \times \mathbf{Q} + \mathbf{P} \times \mathbf{R} \tag{4.11}$$

$$\mathbf{Q} \times \mathbf{P} = -\mathbf{P} \times \mathbf{Q} \tag{4.12}$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j} \tag{4.13}$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \tag{4.14}$$

$$\mathbf{P} \times \mathbf{Q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \tag{4.15}$$

TRIPLE CROSS-SCALAR PRODUCT is defined as

$$(\mathbf{P} \times \mathbf{Q}) \cdot \mathbf{R} \tag{4.16}$$

Useful relations

$$(\mathbf{P}\times\mathbf{Q})\cdot\mathbf{R} = (\mathbf{R}\times\mathbf{P})\cdot\mathbf{Q} = (\mathbf{Q}\times\mathbf{R})\cdot\mathbf{P} \tag{4.17}$$

$$(\mathbf{P}\times\mathbf{Q})\cdot\mathbf{R} = \begin{vmatrix} P_x & P_y & P_z \\ Q_x & Q_y & Q_z \\ R_x & R_y & R_z \end{vmatrix} \tag{4.18}$$

TRIPLE VECTOR PRODUCT is defined as

$$\mathbf{P}\times(\mathbf{Q}\times\mathbf{R}) \tag{4.19}$$

Useful relations

$$\mathbf{P}\times(\mathbf{Q}\times\mathbf{R}) = \mathbf{Q}\cdot(\mathbf{R}\cdot\mathbf{P}) - \mathbf{R}\cdot(\mathbf{P}\cdot\mathbf{Q}) \tag{4.20}$$

DERIVATIVE OF A VECTOR

$$\frac{d\mathbf{P}}{dt} = \dot{\mathbf{P}} = \lim_{\Delta t\to 0}\left(\frac{\Delta\mathbf{P}}{\Delta t}\right) \tag{4.21}$$

Useful relations

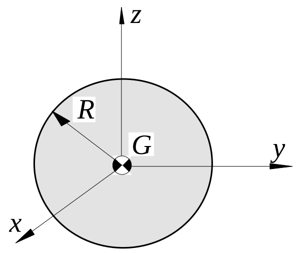
$$\frac{d(\mathbf{P}f)}{dt} = \dot{\mathbf{P}}f + \mathbf{P}\dot{f} \tag{4.22}$$

$$\frac{d(\mathbf{P}\times\mathbf{Q})}{dt} = \dot{\mathbf{P}}\times\mathbf{Q} + \mathbf{P}\times\dot{\mathbf{Q}} \tag{4.23}$$

$$\frac{d(\mathbf{P}\cdot\mathbf{Q})}{dt} = \dot{\mathbf{P}}\cdot\mathbf{Q} + \mathbf{P}\cdot\dot{\mathbf{Q}} \tag{4.24}$$

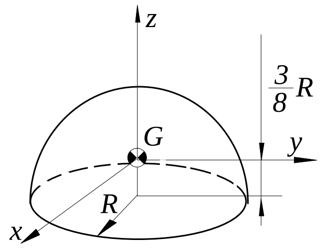
4.2 APPENDIX 2. CENTRE OF GRAVITY, VOLUME AND MOMENTS OF INERTIA OF RIGID BODIES.

Sphere



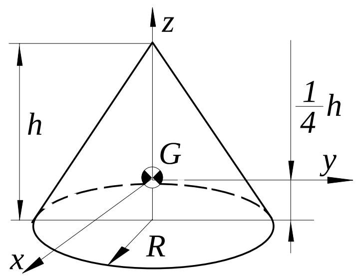
$$V = \frac{4}{3}\pi R^3 \qquad I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}mR^2 \tag{4.25}$$

Hemisphere



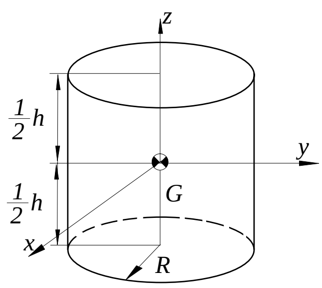
$$V = \frac{2}{3}\pi R^3 \qquad I_{xx} = I_{yy} = 0.259mR^2 \qquad I_{zz} = \frac{2}{5}mR^2 \tag{4.26}$$

Cone



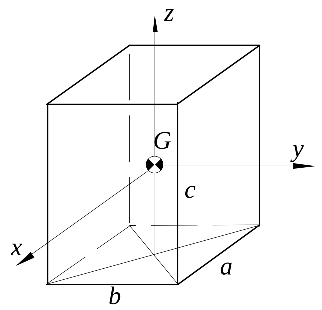
$$V = \frac{1}{3}\pi R^2 h \qquad I_{xx} = I_{yy} = \frac{3}{80}m(4R^2 + h^2) \qquad I_{zz} = \frac{3}{10}mR^2 \tag{4.27}$$

Cylinder



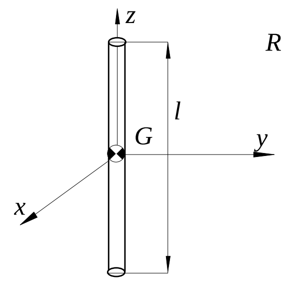
$$V = \pi R^2 h \qquad I_{xx} = I_{yy} = \frac{1}{12}m(3R^2 + h^2) \qquad I_{zz} = \frac{1}{2}mR^2 \tag{4.28}$$

Rectangular block



$$V = abc \qquad I_{xx} = \frac{1}{12}m(b^2 + c^2) \qquad I_{yy} = \frac{1}{12}m(a^2 + c^2) \qquad I_{zz} = \frac{1}{12}m(a^2 + b^2) \tag{4.29}$$

Slender rod



$$V = 0 \qquad I_{xx} = I_{yy} = \frac{1}{12}ml^2 \qquad I_{zz} = 0 \tag{4.30}$$