ME 301

Kinematics & Dynamics of Machines

Class Notes

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ME 301 Kinematics & Dynamics of Machines

Introduction

Kinematics:

Kinema - Greek for motion

Dynamics:

Rigid Body Mechanics Diagram:

Required Math: Geometry, trigonometry, vectors, matrices, calculus

Mechanisms: linkages, cams, gears, gear trains

Analysis vs. Synthesis

- Analysis – determination of position, velocity, acceleration, etc. for a given mechanism
- Synthesis – design of mechanism to do a specific job

Mobility - number of **degrees-of-freedom (dof)**:

- Structure – static, no motion
- Mechanism – 1 dof device with rigid links connected with joints
- Machine – collection of mechanisms to transmit force (input / output)
- Robot – an electromechanical device having greater than 1 dof, programmable for a variety of tasks.

Motion – Translation and Rotation

Planar – all motion is 2D (projected onto a common plane)

Helical - rotation about fixed axis and translation along axis - screw

Spherical - 3D motion; all points in a body moves about a fixed point

Spatial - 3 independent translations and rotations

Joints – Pairing elements

Lower – surface contact

Revolute – pin joint, turning pair

Prismatic – sliding pair

Higher – point or line contact

ball bearing

gears

cam and follower

Link – rigid body

Kinematic chain – number of links connected by joints

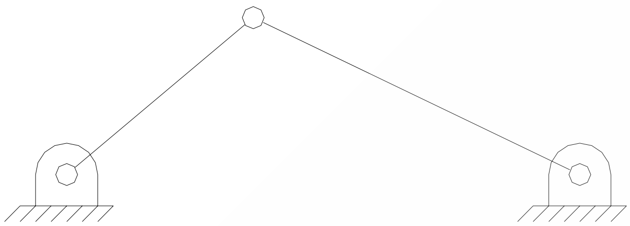
open – serial robot

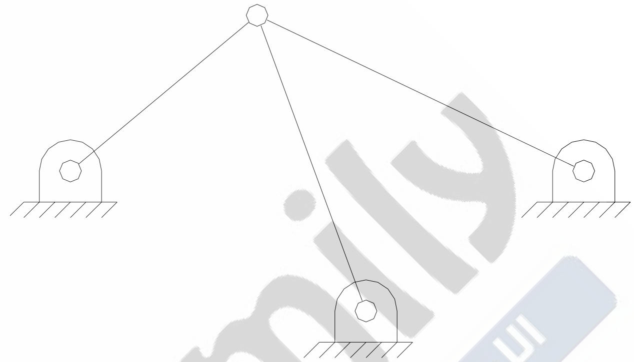
closed – mechanism, parallel robot

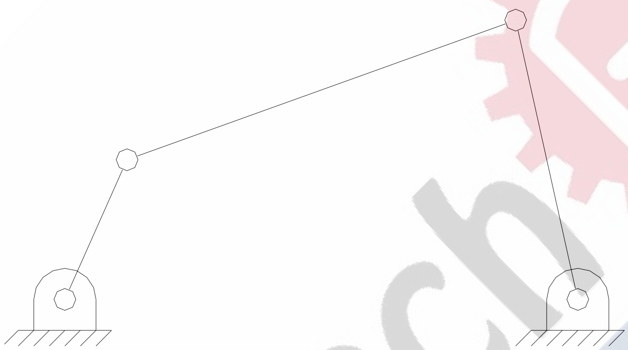
Kinematic Inversion – change which link is fixed – same relative motion, different absolute motion.

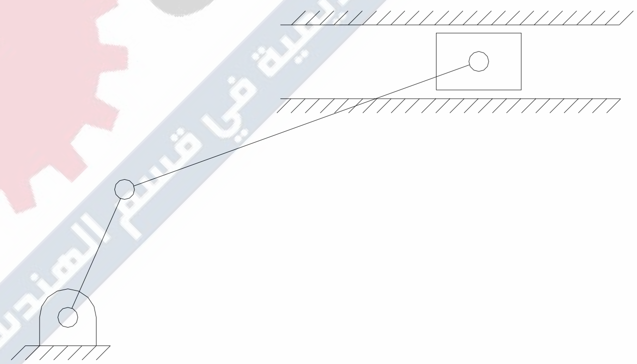
Examples – in class; also see following Atlas

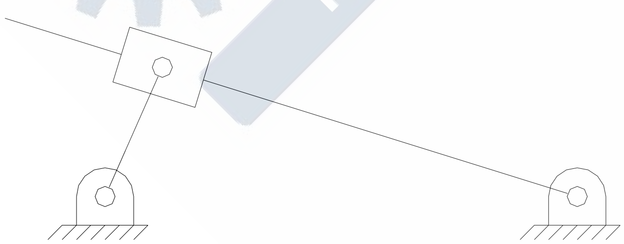
A Brief Atlas of Structures, Mechanisms, and Robots
Dr. Bob

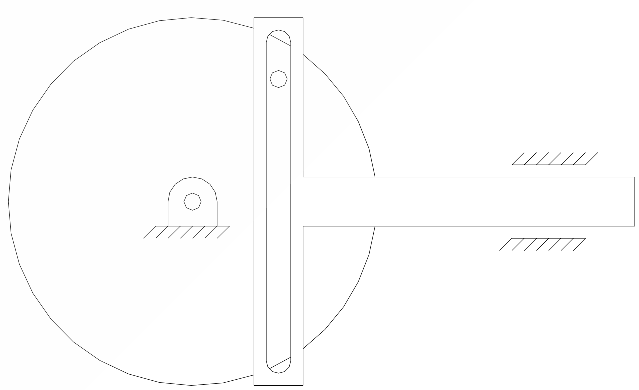

Statically Determinate Structure

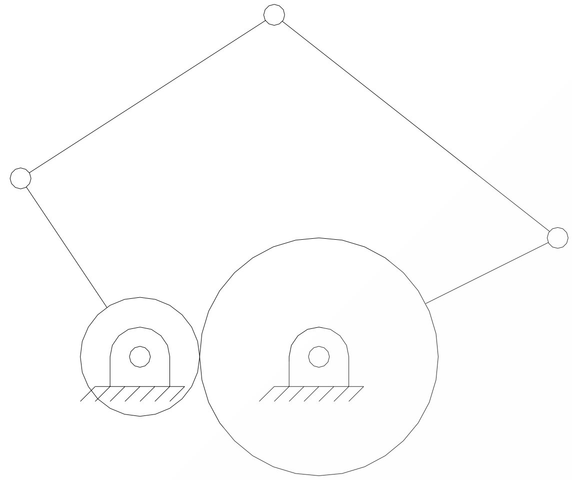

Statically Indeterminate Structure


4-Bar Mechanism

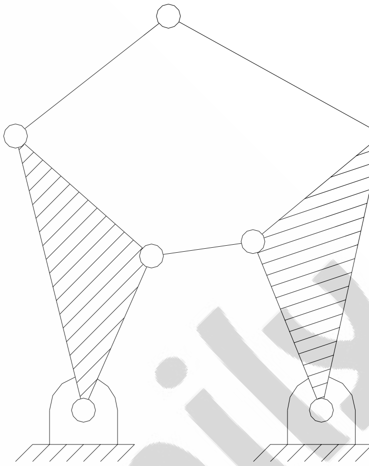

Offset Slider-Crank Mechanism


Inverted Slider-Crank Mechanism

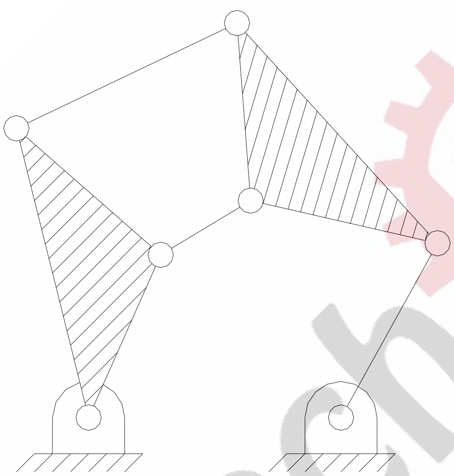

Scotch-Yoke Mechanism



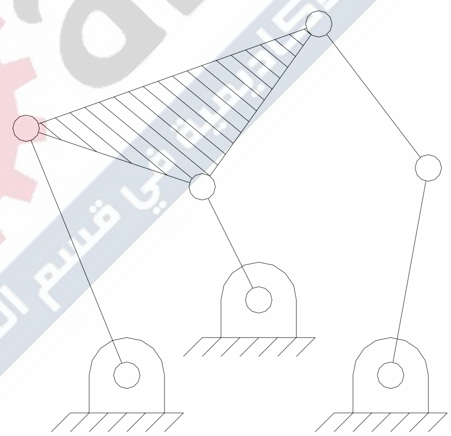
Geared 5-Bar Mechanism



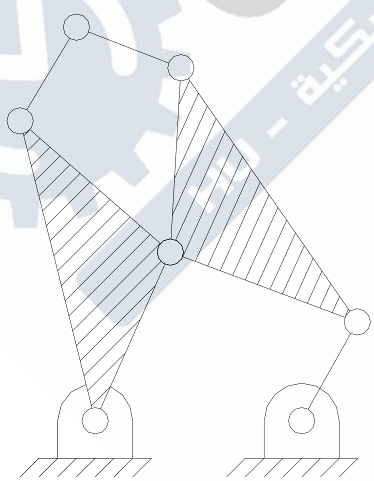
Stephenson I 6-Bar Mechanism



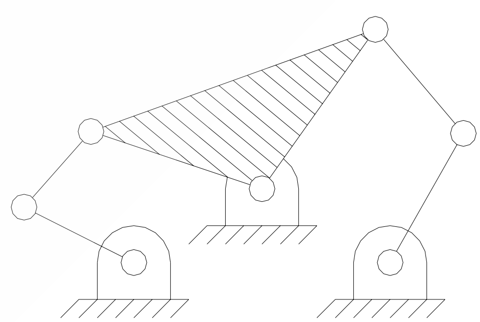
Stephenson II 6-Bar Mechanism



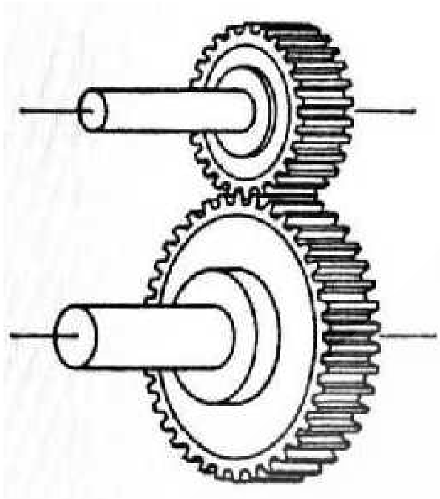
Stephenson III 6-Bar Mechanism



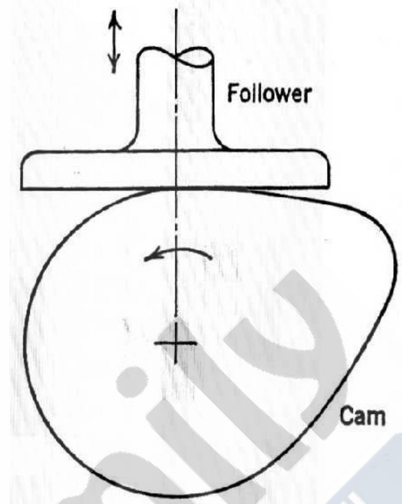
Watt I 6-Bar Mechanism



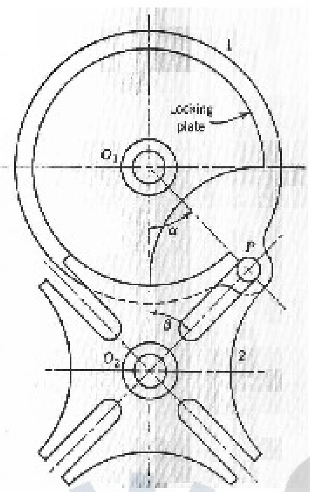
Watt II 6-Bar Mechanism



Spur Gear Mechanism



Cam-and-Follower Mechanism



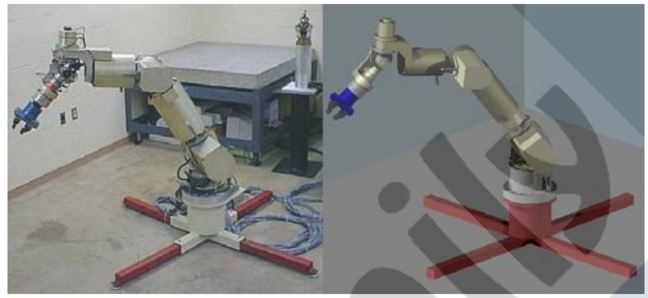
Geneva Wheel Mechanism

Planar 3-dof Robot

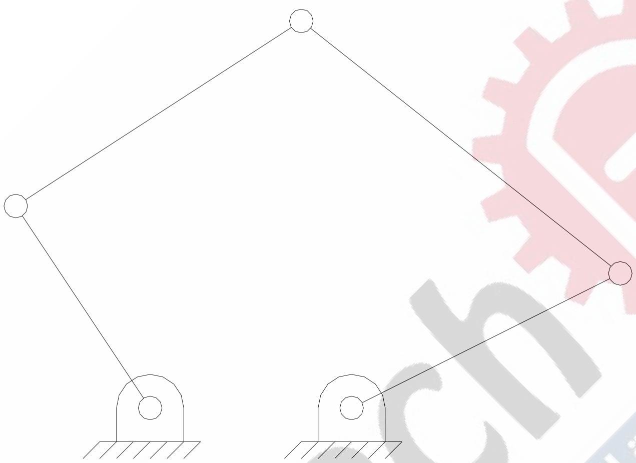
Adept 4-dof SCARA Robot

Mitsubishi 5-dof Robot

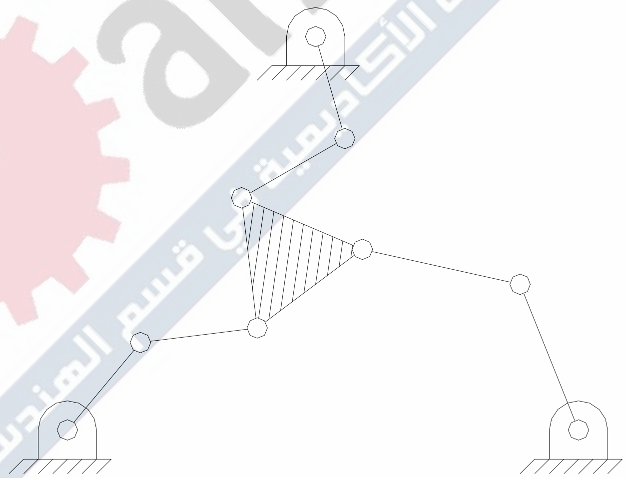
PUMA 6-dof Robot



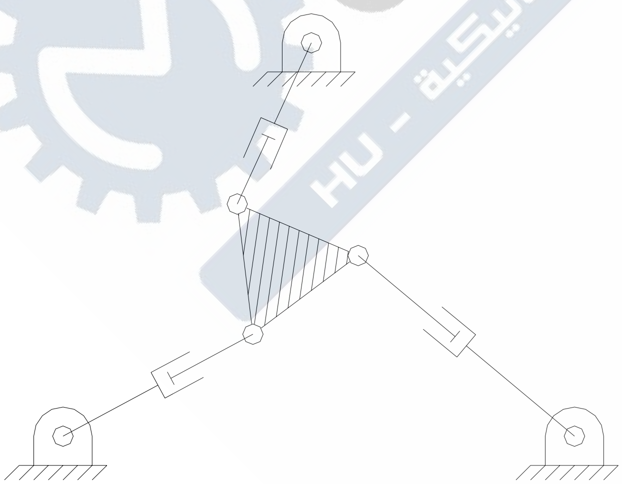
NASA 8-dof ARMII



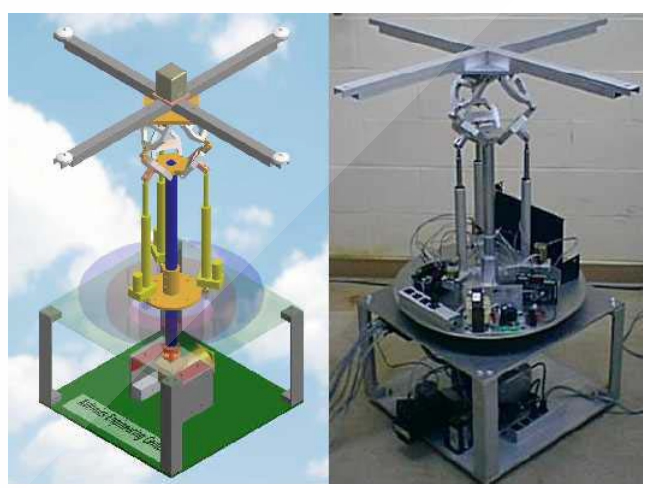
2-dof 5-Bar Parallel Robot



3-dof 3-*RRR* Parallel Robot



3-dof 3-*RPR* Parallel Robot



3-dof Carpal Wrist

Connection to Machine Design

In ME 301 we focus on **kinematics** & **dynamics analysis**, not synthesis (design).

However, the skills gained in this course support general (electro)mechanical design.

Before one can design a machine, the required motion must be satisfied. All design candidates must be analyzed regarding the motion each would provide (position, velocity, and acceleration, both translational and rotational). This requires **kinematics analysis**.

Before one can size the links, joints, bearings, gear box, and actuators (motors) in a machine, the worst-case force and moment loading condition(s) must be known, for statics and dynamics. This requires **dynamics analysis**.

Engineering design is **iterative** by nature: each candidate design must be thoroughly analyzed to determine its performance relative to the design specifications and relative to other design candidates.

This kinematics & dynamics analysis is facilitated using a computer. Without the computer, it is difficult to determine the worst-case loading cases, and over-designed factors of safety may be inefficiently applied.

The goal of ME 301 is to give the student general skills in general matrix/vector-based kinematics and dynamics analysis which may be applied in later classes and later careers.

Matrix-Vector Introduction

Vectors

Arrow in the plane with magnitude and direction. Used to represent position, velocity, acceleration, force. Also, arrow normal to the plane to represent angular velocity, angular acceleration, and torque (moment) vectors (see later in notes).

Cartesian representation:

Polar representation: Magnitude at angle: $\|\underline{P}\| @ \theta$

(*atan2* - quadrant-specific inverse tangent function)

Vector Addition

Vectors add tail-to-head (subtract head-to-tail); express components in same coordinate frame.

Vector Dot Product

Dot product is projection of one vector onto another. Scalar result.

Vector Cross Product

Cross product of two vectors gives a third vector mutually perpendicular to the original two vectors. Vector result.

Direction via right-hand-rule: Put right hand fingers along first vector $\underline{P}_1$, rotate into second vector $\underline{P}_2$; right thumb is direction of $\underline{P}_1 \times \underline{P}_2$.

$\hat{k}$ Vectors

In planar kinematics, angular velocity, angular acceleration, and torque (moment) vectors are arrows along about the $\hat{k}$ axis (the unit direction for the Z axis, perpendicular to the plane). Still has magnitude and direction, but simplifies to a single component with ± sign. We will often represent these $\hat{k}$ vectors by curled arrows in the XY plane.

Example:

$\underline{\omega} = \pm\omega\hat{k}$;

+ ccw (curling in the direction of the right hand fingers)
– cw (curling in the opposite direction of the right hand fingers)

Vector Examples

$$P_1 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \qquad P_2 = \begin{Bmatrix} 3 \\ 2 \end{Bmatrix}$$

Addition: $P_1 + P_2 =$

Dot Product: $P_1 \bullet P_2 =$

Cross Product: $P_1 \times P_2 =$

Matrices

Matrix: m x n array of numbers, where m is the number of rows and n in the number of columns.

$$[A]=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Used to simplify and standardize the solution of n linear equations in n unknowns (where $m=n$). Used in velocity, acceleration, and dynamics analysis linear equations (not used in position which is a non-linear solution).

Special Matrices

Square ($m=n=3$) $$[A]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Diagonal $$[A]=\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Identity $$[I]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transpose $$[A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Symmetric $$[A] = [A]^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Column Vector (3x1 matrix) $$\{X\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Row Vector (1x3 matrix) $$\{X\}^T = \{x_1 \quad x_2 \quad x_3\}$$

Matrix Addition Just add up like terms

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Matrix Multiplication with Scalar Just multiply each term

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

Matrix Multiplication $[C]=[A][B]\neq[B][A]$

Row, Column indices have to line up as follows:

$$[C]=[A][B]$$
$$(mxn)\equiv(mxp)(pxn)$$

That is, the number of columns in the left-hand matrix must equal the number of rows in the right-hand matrix; if not, the multiplication is undefined and cannot be done! Multiplication proceeds by multiplying and adding terms along the rows of the left-hand matrix and down the columns of the right-hand matrix: (use your index fingers from the left and right hands):

Example: $$[C]=\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{Bmatrix} g \\ h \\ i \end{Bmatrix}=\begin{Bmatrix} ag+bh+ci \\ dg+eh+fi \end{Bmatrix}$$

$$(2x1)\equiv(2x3)(3x1)$$

note the inner indices (p=3) must match, as stated above and the dimension of the result is the outer indices, i.e. $2x1$.

Matrix Multiplication Examples

$$[A]=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} \qquad [B]=\begin{bmatrix}7 & 8\\9 & 8\\7 & 6\end{bmatrix}$$

$$[C]=[A][B]$$

$$=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}\begin{bmatrix}7 & 8\\9 & 8\\7 & 6\end{bmatrix}$$

$$=\begin{bmatrix}7+18+21 & 8+16+18\\28+45+42 & 32+40+36\end{bmatrix}=\begin{bmatrix}46 & 42\\115 & 108\end{bmatrix}$$

$$(2x2)\equiv(2x3)(3x2)$$

$$[D]=[B][A]$$

$$=\begin{bmatrix}7 & 8\\9 & 8\\7 & 6\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$$

$$=\begin{bmatrix}7+32 & 14+40 & 21+48\\9+32 & 18+40 & 27+48\\7+24 & 14+30 & 21+36\end{bmatrix}=\begin{bmatrix}39 & 54 & 69\\41 & 58 & 75\\31 & 44 & 57\end{bmatrix}$$

$$(3x3)\equiv(3x2)(2x3)$$

Matrix Inversion

Matrix "division": given $[C]=[A][B]$, solve for $[B]$

$$[C]=[A][B]\Rightarrow$$

$$\begin{aligned}[A]^{-1}[C]&=[A]^{-1}[A][B]\\&=[I][B]\\&=[B]\end{aligned}$$

$$\Rightarrow [B]=[A]^{-1}[C]$$

Matrix $[A]$ must be square to invert.

$$[A][A]^{-1}=[A]^{-1}[A]=[I]$$

where $[I]$ is the identity matrix, the matrix "1". To calculate the matrix inverse:

$$[A]^{-1}=\frac{[\text{Adjoint}(A)]}{|A|}$$

where: $|A|$ Determinant of $[A]$

$$[\text{Adjoint}(A)]=[\text{Cofactor}(A)]^T$$

Cofactor(A) $\quad a_{ij}=(-1)^{i+j}M_{ij}$

Minor M_{ij} is the determinant of the submatrix with row i and column j removed.

System of Linear Equations

We can solve n linear equations in n unknowns with the help of a matrix. For n=3:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

Using matrix multiplication (backwards), this is written as:

$$[A]\{x\} = \{b\}$$

where:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \text{(known coefficients)}$$

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \qquad \text{(unknowns to be solved)}$$

$$\{b\} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \qquad \text{(known right-hand sides)}$$

Unique solution $\{x\} = [A]^{-1}\{b\}$ only if [A] has full rank. If not, $|A| = 0$ and the inverse of matrix [A] is undefined (dividing by zero).

Matrix Example

Solution of simultaneous linear equations.

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 6x_1 + 4x_2 &= 14 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 1 & 2 \\ 6 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 14 \end{Bmatrix}$$

$$[A] = \begin{bmatrix} 1 & 2 \\ 6 & 4 \end{bmatrix} \qquad \{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad \{b\} = \begin{Bmatrix} 5 \\ 14 \end{Bmatrix}$$

$$\{x\} = [A]^{-1}\{b\}$$

$|A| = 1(4) - 2(6) = -8$ Determinant non-zero; unique solution!

$$[A]^{-1} = \frac{1}{|A|}\begin{bmatrix} 4 & -2 \\ -6 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/4 \\ 3/4 & -1/8 \end{bmatrix}$$

check: $[A][A]^{-1} = [A]^{-1}[A] = [I_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} -1/2 & 1/4 \\ 3/4 & -1/8 \end{bmatrix} \begin{Bmatrix} 5 \\ 14 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$ Answer.

check: Plug answer into original equations and compare to the $\{b\}$ we need to get.

Vector and Matrix Matlab Examples

```
P1 = [1;2;0];           % Define two vectors
P2 = [3;2;0];
sum1 = P1+P2;           % Vector addition
sum2 = P2+P1;
dot1 = dot(P1,P2);      % Vector dot product
dot2 = dot(P2,P1);
cross1 = cross(P1,P2); % Vector cross product
cross2 = cross(P2,P1);


A = [1  2;6  4];        % Define a matrix and vector
b = [5;14];
dA = det(A);            % Calculate determinant of A
invA = inv(A);          % Calculate the inverse of A
x = invA*b;             % Solve linear equations
x1 = x(1);              % Extract answers
x2 = x(2);
A*x                     % Check answer - should be b
```

Matlab Introduction

***Mat**rix **lab**oratory*

Control systems simulation and design software. Very widespread in other fields. Introduction to basics, programming, plots, animation, matrices, vectors. Based on C language, programming is vaguely C-like, but much simpler to use. Sold by Mathworks (http://www.mathworks.com).

Can buy student version software and manual for about the price of one textbook (can use it for many classes!). ENT college has a Matlab license; it is installed in most computer labs.

Double-click on Matlab icon to get started. Type

>>demo

to get a comprehensive overview of Matlab including built-in functions. Try all the categories under Matlab first; you can ignore Toolboxes, Simulink, and Stateflow for now. (Exception: there is Symbolic Math under Toolboxes for the adventurous student!).

Type in commands (such as the Vector/Matrix examples given earlier) at the Matlab prompt >>. Press <Enter> to see result or ; <Enter> to suppress result.

Recommended operation mode: **<u>m-files</u>**. Put your sequence of Matlab statements in an ASCII file ***name.m*** (create a file with the beautiful Matlab Editor/Debugger - this is color-coordinated, tab-friendly, with parentheses alignment help and debugging capabilities). A % indicates a comment. One basic way to run your program is to hit the 'save and run' button on the editor toolbar.

Alternative: at the >> prompt type the M-File name ***name***, without the ***.m***, assuming your file is in the search path. Matlab language is interpretive and executes line-by-line. Use the ; at the end of statements to suppress intermediate results. If you use this suppression, the variable name still holds the resulting value(s) – just type the variable name at the prompt after the program runs to see the value(s). If there is a syntax or programming logic error, it will give a message at the bad line and then quit. Type:

>>*who*

to show you what variables you have defined;

>>*whos*

will show the variables, plus their matrix dimensions (scalar, vector array, or matrix), very useful for debugging. Plus, after running a file, place the cursor over different variables in the M-File inside the Editor/Debugger to see the values! On-line help is generally great:

>>*help*

Example m-files (given on the following two pages)

1) *MatEx1.m*: Input, programming, plots, animation.

2) *MatEx2.m*: Matrix and vector definition, multiplication, transpose, and solution of linear equations.

```
%------------------------------------------------------------------
%  Matlab Example Code 1:  MatEx1.m
%     Matrix, Vector examples
%        Dr. Bob, ME 301
%------------------------------------------------------------------

clc; clear;    %  Clear the cursor and clear any previously defined variables


%
%  Matrix and Vector definition, multiplication, and transpose
%

A1  = [1  2 3; ...    %  Define 2x3 matrix [A1] (... is continuation line)
       1 -1 1];
x1  = [1;2;3];        %  Define 3x1 vector {x1}

v  = A1*x1;           %  2x1 vector {v} is the product of [A1] times {x1}

A1T = A1';            %  Transpose of matrix [A1]
vT = v';              %  Transpose of vector {v}


%
%  Solution of linear equations Ax=b
%

A2    = [1  2  3; ...    %  Define matrix [A2] to be a 3x3 coefficient matrix
         1 -1  1; ...
         8  2 10];

b     = [3;2;1];      %  Define right-hand side vector of knowns {b}

detA2  = det(A2);     %  First check to see if det(A) is near zero

x2     = inv(A2)*b;   %  Calculate {x2} to be the solution of Ax=b by inversion

check = A2*x2;        %  Check results;

z     = b - check;    %  Better be zero!


%
%  Display the user-created variables (who), with dimensions (whos)
%

who
whos


%
%     Display some of the results
%

v
x2
z
```

```matlab
%---------------------------------------------------------------
%  Matlab Example Code 2:  MatEx2.m
%     Menu, Input, FOR loop, IF logic, Animation, and Plotting
%        Dr. Bob, ME 301
%---------------------------------------------------------------

clc;  clear;   %  Clear the cursor and clear any previously defined variables

r  = 1;  L  = 2;  DR = pi/180;                     %  Constants

%
%  Input
%
anim = menu('Animate Single Link?','Yes','No')     %  Menu to screen
the = input('Enter [th0, dth, thf] (deg):  ')      %  User types input
th0 = the(1)*DR; dth = the(2)*DR; thf = the(3)*DR; %  Initial, delta, final thetas
th  = [th0:dth:thf];                               %  Assign theta array
N   = (thf-th0)/dth + 1;                           %  Number of iterations for loop

%
%  Animate single link
%
if anim == 1                                       %  Animate if user wants to
   figure;                                         %  Give a blank graphics window
   for i = 1:N;                                    %  For loop to animate
      x2 = [0     L*cos(th(i))];                   %  Single link coordinates
      y2 = [0     L*sin(th(i))];
      plot(x2,y2); grid;                           %  Animate to screen
      set(gca,'FontSize',18);
      xlabel('\itX (\itm)'); ylabel('\itY (\itm)');
      axis('square'); axis([-2 2 -2 2]);           %  Define square plot limits
      pause(1/4);                                  %  Pause to see animation
      if i==1                                      %  Pause to maximize window
         pause;                                    %  User hits Enter to continue
      end
   end
end

%
%  Calculate circle coordinates and cosine function
%
xc = r*cos(th);                              %  Circle coordinates
yc = r*sin(th);
f1 = cos(th);                                %  Cosine function of theta
f2 = sin(th);                                %   Sine  function of theta

%
%  plots
%
figure;                                      %  Co-plot cosine and sine functions
plot(th/DR,f1,'r',th/DR,f2,'g'); grid; set(gca,'FontSize',18);
legend('Cosine','Sine');
axis([0 360 -1 1]); title('Functions of \it\theta');
xlabel('\it\theta (\itdeg)'); ylabel('Functions of \it\theta');

figure;                                      %  Plot circle
plot(xc,yc,'b'); grid; set(gca,'FontSize',18);
axis(['square']); axis([-1.5 1.5 -1.5 1.5]); title('Circle');
xlabel('\itX (\itm)'); ylabel('\itY (\itm)');
```

Mobility

Mobility:

Degrees-of-freedom (dof):

How many dofs does an unconstrained planar link have?

What is the effect of constraining that link with a revolute joint?

Grubler's Criterion: Planar Jointed Devices

Where: M is the mobility
N is the total # of links, including ground
J_1 is the number of one-degree-of-freedom joints
J_2 is the number of two-degree-of-freedom joints

One-degree-of-freedom joints:

Revolute

Prismatic

Two-degree-of-freedom joints (all have rolling and sliding):

Cam joint

Gear joint

Slotted-pin joint

Caution: *m* links joining at one revolute location, must count *m*-1 joints!

Caution: must count ground link (its freedom is subtracted in formula with *n*-1.

Planar mechanical device classification:

$M > 1$

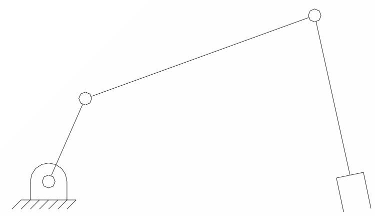
$M = 1$

$M = 0$

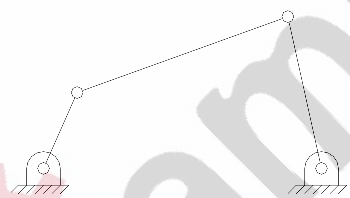
$M < 0$

Planar Mobility Examples:

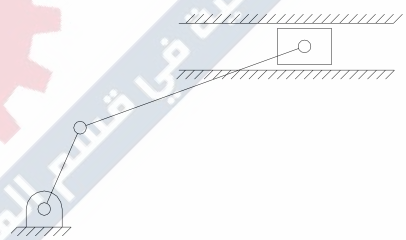
1) 3-link serial robot



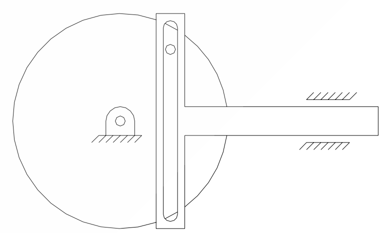
2) 4-bar linkage



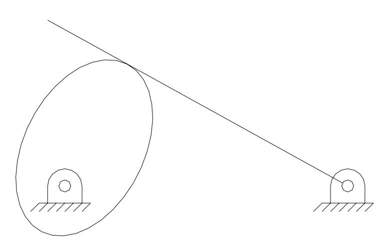
3) Slider-crank linkage

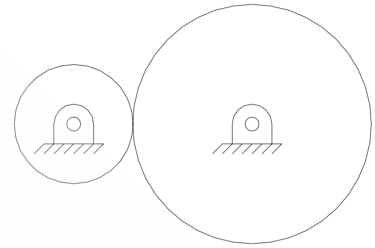


4) Scotch Yoke mechanism

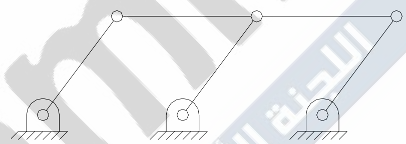


5) Cam and follower

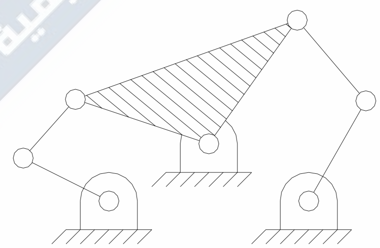




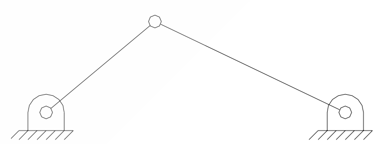
6) Gear pair



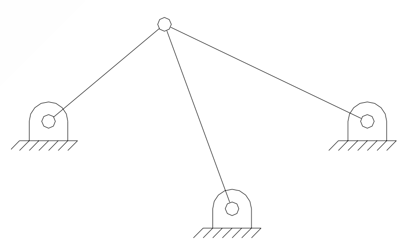
7) 4-bar linkage with parallel link



8) Watt 6-bar linkage

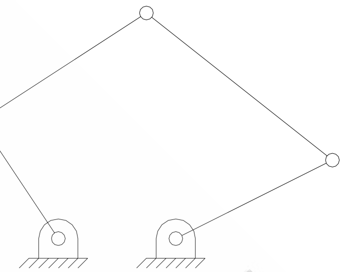


9) Statically-determinate structure

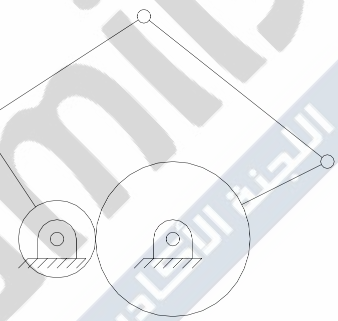


10) Statically-indeterminate structure

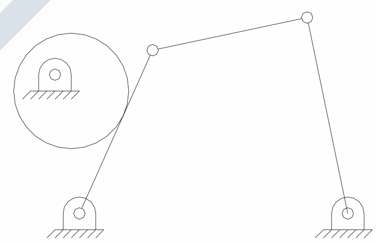
11) 5-bar linkage



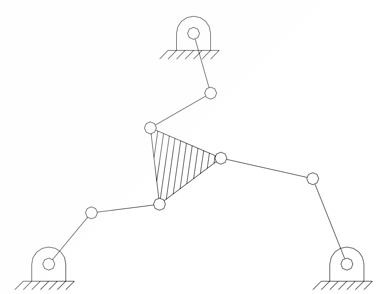
12) Geared 5-bar linkage



13) Cam-modulated 4-bar linkage



14) 3-RRR parallel robot



Four-Bar Mechanism Position Analysis

Position (Displacement) Analysis: determination of relative orientation/ position of links in a mechanism. Required for testing motion of a synthesized mechanism. Also required for further analysis: velocity, acceleration, dynamics, forces.

Generic Mechanism Position Analysis Statement: Given the mechanism and one dof of position input, calculate the position unknowns.

Four-bar Mechanism Position Analysis

Step 1. Draw the **Kinematic Diagram:**

r_1 – fixed ground link θ_1 – ground link angle

r_2 – input link θ_2 – input angle

r_3 – coupler link θ_3 – coupler angle

r_4 – output link θ_4 – output angle

All angles measured in right-hand sense from horizontal to link.

Step 2. **State the problem:**

Step 3. Draw the **Vector Diagram**. Define all angles in positive sense, measured from the right horizontal to the link vector (tail-to-head). Don't try to force acute angles; the relationships we can see so easily in the first quadrant hold for all four quadrants:

$$\underline{P} = \begin{Bmatrix} L\cos\theta \\ L\sin\theta \end{Bmatrix} \text{; good for all } \theta.$$

Vector Diagram:

Step 4. Derive the **Vector-Loop-Closure Equation**. Start at one point, add vectors tail-to-head until reach a second point. Write equation by starting and ending at same points, but choosing a different path.

<u>Step 5.</u> Write <u>***XY* Components**</u> for Vector-Loop-Closure Equation. Break one vector equation into its two scalar components (*X* and *Y*):

<u>Step 6.</u> <u>**Solve for the Unknowns**</u> from the *XY* Equations. Two coupled nonlinear equations in the two unknowns θ_3, θ_4. Isolate and eliminate θ_3 and solve for θ_4. Then go back to find θ_3.

Square and add:

This equation has the form:

Solve using the tangent half angle substitution (Text Equation 4.9):

$$t = \tan\left(\frac{\theta_4}{2}\right) \qquad \cos\theta_4 = \frac{1-t^2}{1+t^2} \qquad \sin\theta_4 = \frac{2t}{1+t^2}$$

We converted a complicated coupled transcendental set of equations into a quadratic polynomial. Much easier to solve (but we doubled the order of the equation!).

Two solutions for θ_4:

With factor two, no need to use the *atan2* function.

Why two solutions? (Graphically demonstrate the two branches.)

What if $E^2 + F^2 - G^2 < 0$? Imaginary solution, physically means the mechanism cannot assemble for that input angle. See section on Grashof's Law.

Go back to find θ_3, one for each solution branch. Go back to original two *XY* scalar equations.

Use ratio of *Y* to *X* equations:

Show graphical interpretation: $\theta_3 = \tan^{-1}\left(\dfrac{B_Y - A_Y}{B_X - A_X}\right)$

The basic four-bar mechanism position analysis problem is now solved. Now that we know the angular unknowns, we can find the translational **position of any point** on the mechanism, e.g. coupler point *C*:

Four-bar mechanism transmission angle: Transmission angle μ: relative angle between coupler and output links. Measure of mechanical advantage of mechanism; 90° is ideal; 0,180° zero transmission; as a rule of thumb, the absolute value of μ should remain in the range $40^\circ < \mu < 140^\circ$ for good transmission in a mechanism. By geometry:

Four-Bar Mechanism Position Analysis: Term Example 1

Given
$$\begin{matrix} r_1 = 11.18 \\ r_2 = 3 \\ r_3 = 8 \\ r_4 = 7 \end{matrix} \quad in \qquad \begin{matrix} r_1 = 0.284 \\ r_2 = 0.076 \\ r_3 = 0.203 \\ r_4 = 0.178 \end{matrix} \quad m$$

and $\theta_1 = 10.3^\circ$ (Ground link is 11" over and 2" up). Also given $R_{C/A} = 5$ (in) and $\delta_3 = 36.9^\circ$ for the coupler link point of interest.

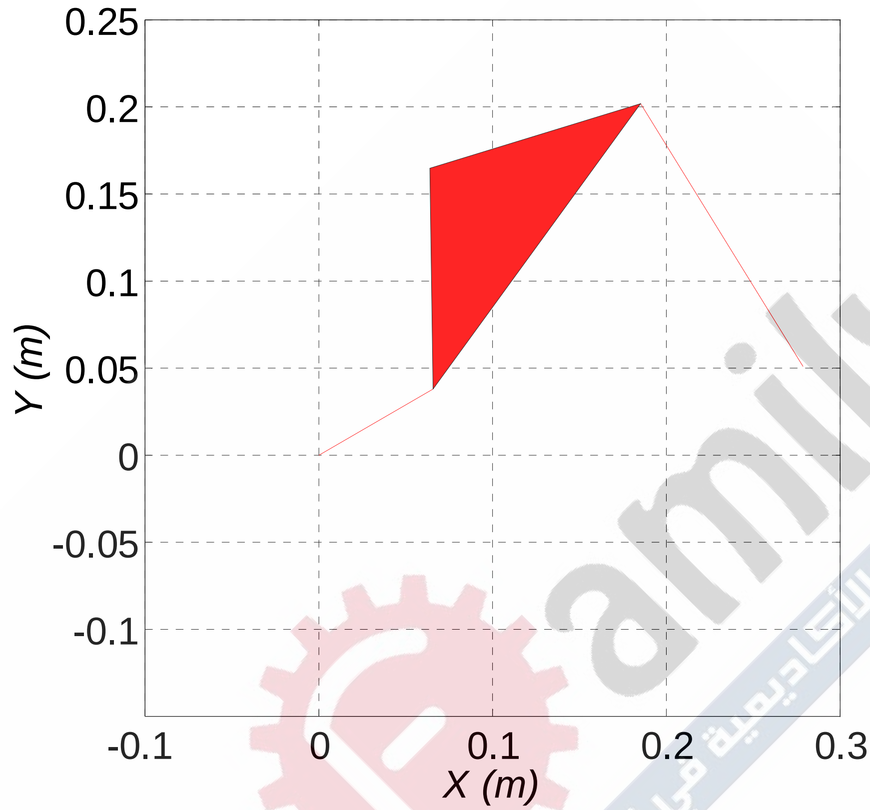
Snapshot Analysis (one input angle)

Given this mechanism and $\theta_2 = 30^\circ$, calculate θ_3, θ_4, μ, and $\underline{P}_C$ for both branches. Results:

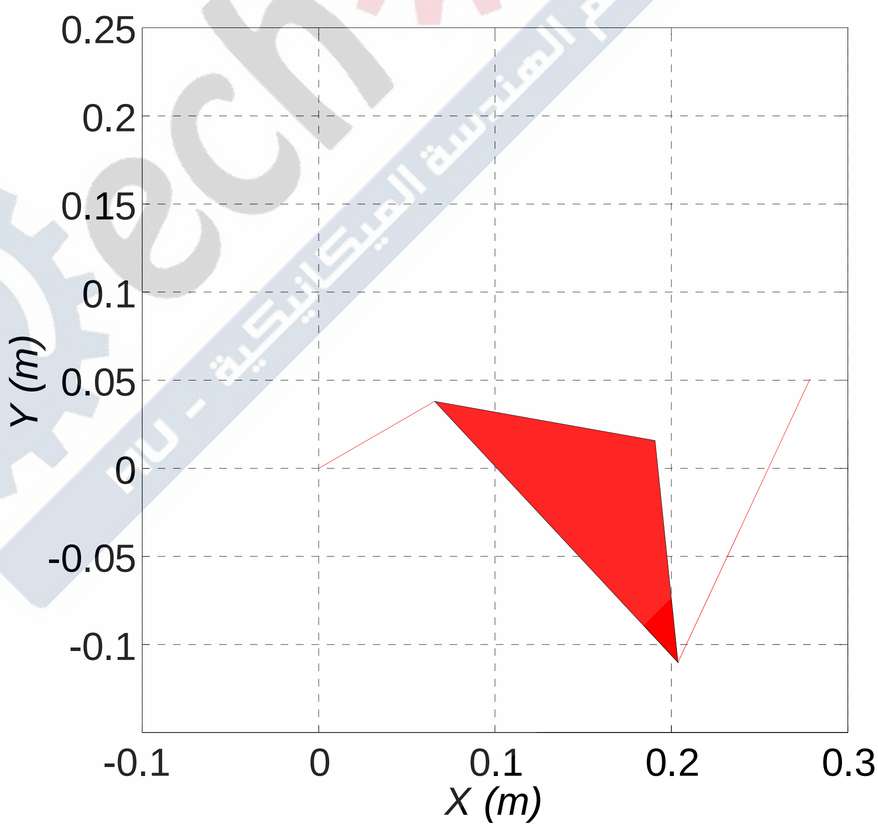
$$E = 0.076$$
$$F = 0.005$$
$$G = 0.036$$

Branch	t	θ_3	θ_4	μ	$\underline{P}_C$
Open	1.79	53.8°	121.7°	67.9°	0.06, 0.16
Crossed	-1.57	-47.0°	-114.9°	67.9°	0.19, 0.02

These two branch solutions are demonstrated in the figures on the following page. We use the SI system (m). Note μ is identical for both branches due to the conventions presented earlier.



4-bar Example Snapshot, Open Branch



4-bar Example Snapshot, Crossed Branch

Graphical Solution: The 4-bar position analysis may be solved *graphically*, merely by drawing the mechanism and determining the mechanism closure. This is an excellent method to validate your computer results at a given snapshot.

- Draw the known ground link (points O_2 and O_4).
- Draw the given input link 2 length at the given angle (point A).
- Draw a circle of radius r_3, centered at point A.
- Draw a circle of radius r_4, centered at point O_4.
- These circles intersect in general in two places.
- Connect the two branches and measure the unknown values.

Graphical Solution Figure:

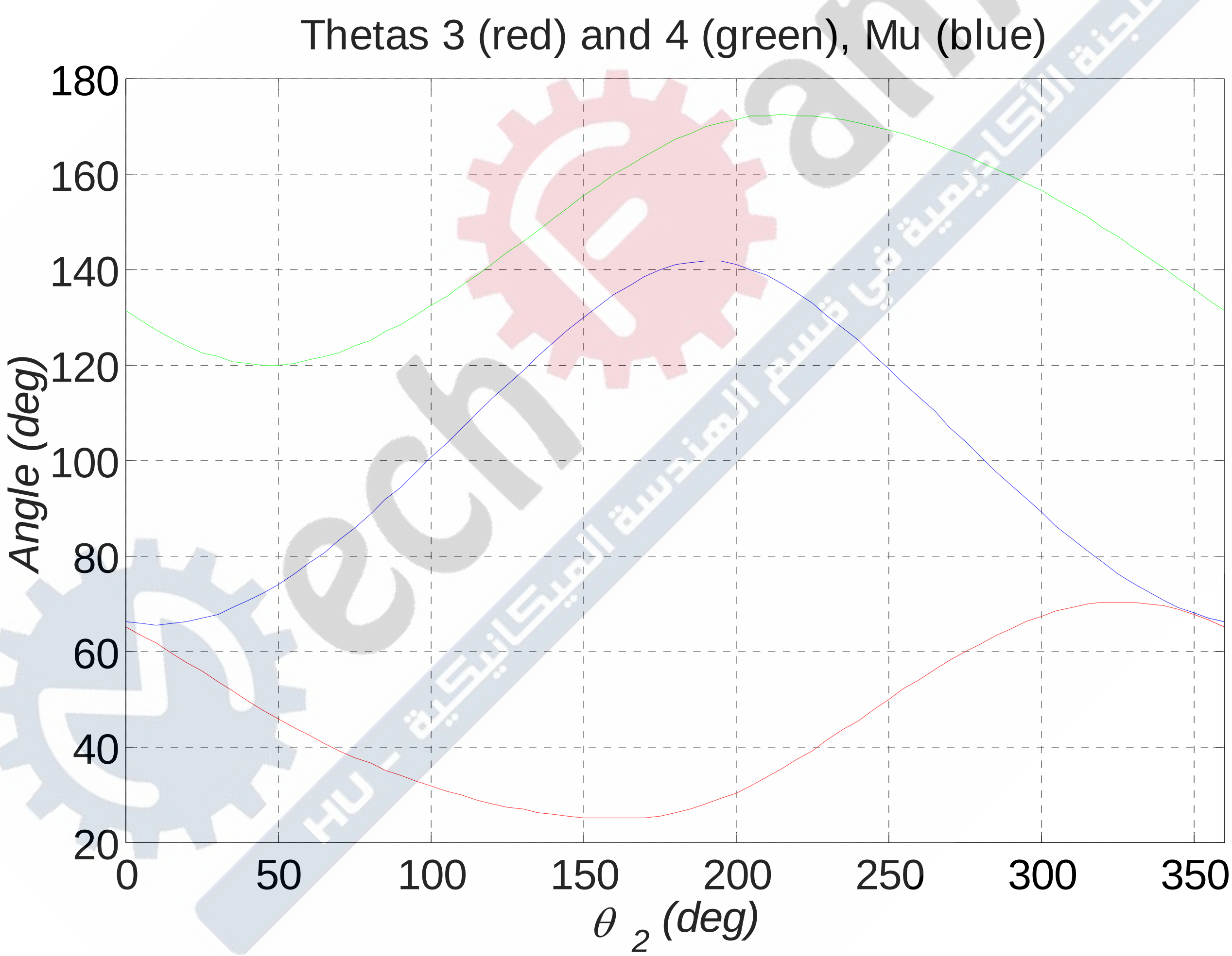
4-bar Snapshot Matlab code:

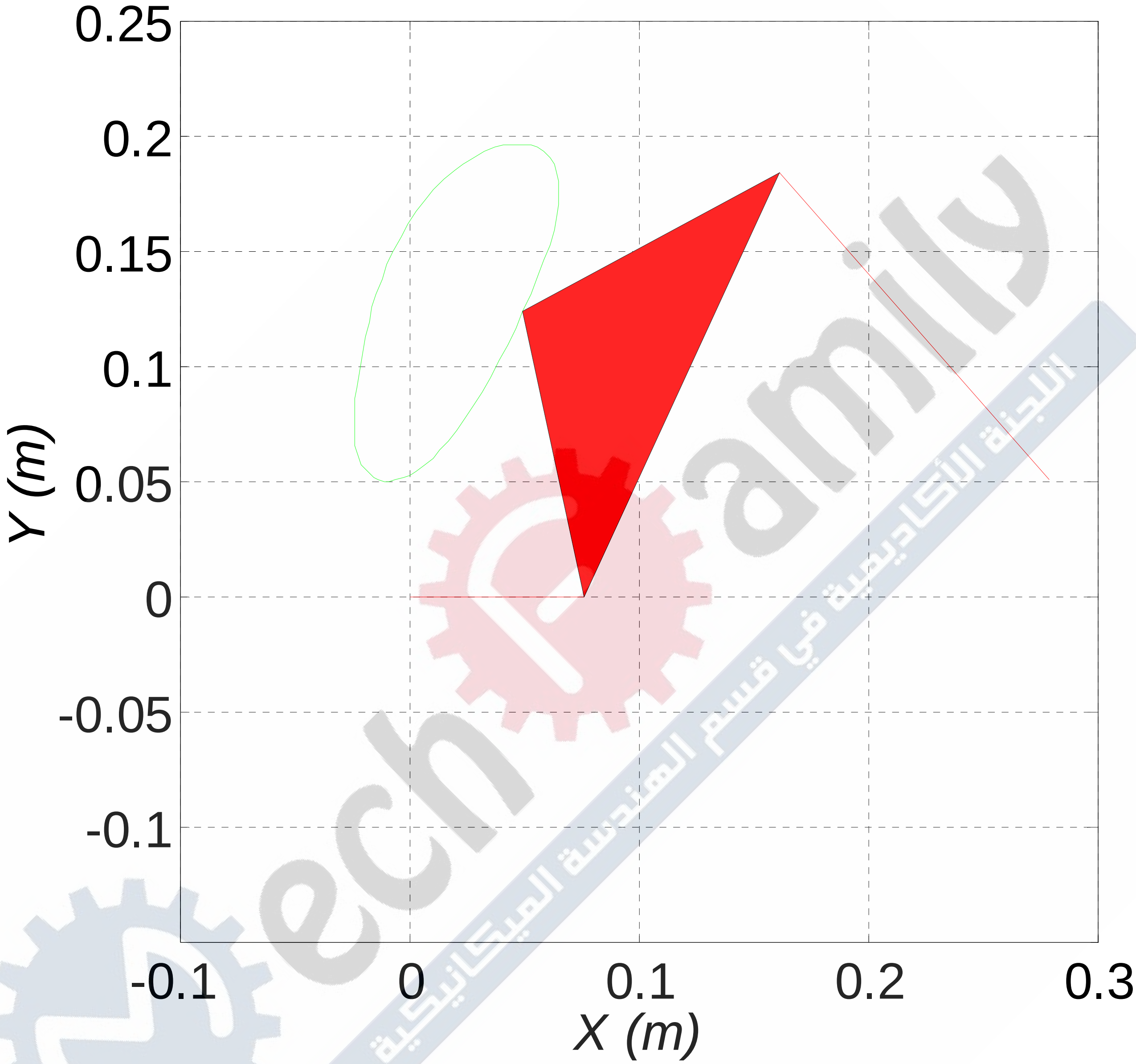
This program solves the 4-bar position analysis problem for both branches given a single θ_2. The results are drawn to the screen.

```
%---------------------------------------------------------------
%  4-bar linkage snapshot position analysis - both branches
%     Fbarplec.m, with graphical output,  Dr. Bob, ME 301
%---------------------------------------------------------------
clc; clear;    %  Clear cursor and clear previously defined variables

%  Inputs
DR = pi/180;
R = input('Enter [r1, r2, r3, r4, rca, th1, th2, del3] (m and deg):  ');
r1 = R(1); r2 = R(2); r3 = R(3); r4 = R(4); rca = R(5);
th1 = R(6)*DR; th2 = R(7)*DR; del3 = R(8)*DR;      %  Change degrees to radians
r1x = r1*cos(th1); r1y = r1*sin(th1);

%  Position analysis:  theta4
E =  2*r4*(r1*cos(th1) - r2*cos(th2));
F =  2*r4*(r1*sin(th1) - r2*sin(th2));
G =  r1^2 + r2^2 - r3^2 + r4^2 - 2*r1*r2*cos(th1-th2);
t(1)  = (-F + sqrt(E^2 + F^2 - G^2)) / (G-E);       %  Crossed Branch
t(2)  = (-F - sqrt(E^2 + F^2 - G^2)) / (G-E);       %   Open   Branch
th4(1)  = 2*atan(t(1));
th4(2)  = 2*atan(t(2));

%  th3, coupler point, transmission angle; calculate for both branches
for i = 1:2,
   ax = r2*cos(th2);                            %  theta3
   ay = r2*sin(th2);
   bx = r4*cos(th4(i)) + r1x;
   by = r4*sin(th4(i)) + r1y;
   th3(i)  = atan2(by-ay,bx-ax);
   bet  = th3(i) + del3;                        %  coupler point
   pcx(i)  = r2*cos(th2) + rca*cos(bet);
   pcy(i)  = r2*sin(th2) + rca*sin(bet);
   mu(i)  = abs(th4(i)-th3(i));                 %  transmission angle
end

%     Plot 4-bar position results
for i = 1:2,
   x2 = [0            r2*cos(th2)];                 %  Coords of link 2
   y2 = [0            r2*sin(th2)];
   x3 = [r2*cos(th2)  r1x+r4*cos(th4(i)) pcx(i)];  %  Coords of link 3
   y3 = [r2*sin(th2)  r1y+r4*sin(th4(i)) pcy(i)];
   x4 = [r1x          r1x+r4*cos(th4(i))];          %  Coords of link 4
   y4 = [r1y          r1y+r4*sin(th4(i))];
   figure;
   plot(x2,y2,'r',x4,y4,'r'); patch(x3,y3,'r');
   axis('square'); set(gca,'FontSize',18);
   xlabel('\itX (\itm)'); ylabel('\itY (\itm)');
   axis([-0.1 0.3 -0.15 0.25]); grid;
end
```

Full-Range-Of-Motion (F.R.O.M.) Analysis

A more meaningful result from position analysis is to report the position analysis unknowns for the entire range of mechanism motion. The first plot gives θ_3 (red), θ_4 (green), and μ (blue), all *deg*, for all $0^\circ \le \theta_2 \le 360^\circ$, for Term Example 1, open branch only. The second plot gives the coupler point location for this branch, plotting P_{CY} vs. P_{CX}.





4-bar Example Snapshot, Open Branch Coupler Curve

Trigonometric Uncertainty

Return to θ_3 solution: *XY* scalar equations:

$$r_3 c_3 = r_1 c_1 + r_4 c_4 - r_2 c_2$$
$$r_3 s_3 = r_1 s_1 + r_4 s_4 - r_2 s_2$$

Since θ_4 has been solved, why not calculate θ_3 using *Y* equation?:

e.g. $\theta_3 = \sin^{-1}(0.5)$; figure:

Problem: *inverse sine function* is **double-valued**; for each θ_4 there are two possible solutions, only one of which is correct! Why not calculate θ_3 using *X* equation? Inverse cosine has a similar problem;

e.g. $\theta_3 = \cos^{-1}\left(\sqrt{3}/2\right)$; figure:

Problem: *inverse cosine function* is **double-valued**; for each θ_4 there are two possible solutions, only one of which is correct!

So we must use information from both sine and cosine (i.e. both *X* and *Y* equations) - this suggests using the tangent (as we did earlier in the θ_3 solution):

$$\theta_3 = \tan^{-1}\left(\frac{r_1 s_1 + r_4 s_4 - r_2 s_2}{r_1 c_1 + r_4 c_4 - r_2 c_2}\right)$$

e.g. $\theta_3 = \tan^{-1}\left(1/\sqrt{3} \right)$; figure:

Problem: the plain *atan inverse tangent function* is still **double-valued!**; for each θ_4 there are two possible solutions, only one of which is correct! Solution: use the **quadrant-specific** inverse tangent function *atan2*. Input to this function is both a numerator and denominator; the function has built-in logic to determine the correct quadrant for the angle answer, given the signs ± of the numerator and denominator. The plain *atan* function takes a single quotient input; hence this sign information is lost and the true quadrant is unknown. **No uncertainty** with *atan2*:

e.g. $\theta_3 = a\tan 2\left(+\frac{1}{2}, +\frac{\sqrt{3}}{2} \right) =$

$\theta_3 = a\tan 2\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2} \right) =$

$$\theta_3 = A\tan 2\left(r_1 s_1 + r_4 s_4 - r_2 s_2, r_1 c_1 + r_4 c_4 - r_2 c_2\right)$$

Now, having just cleared up this **Trigonometric Uncertainty**, we already have an exception in the θ_4 tangent half-angle solution:

$$\theta_4 = 2\tan^{-1}(t)$$

(there are two branches, one for each *t* value; only showing one here.)

With the 2 multiplying the inverse tangent result, it doesn't matter whether we use *atan* or *atan2* since the final answer will come to the same angle. Example:

For $\frac{\theta_4}{2} = \tan^{-1}\left(1/\sqrt{3} \right)$, from before, we don't know if the solution is

$$\frac{\theta_4}{2} = 30^\circ \text{ or}$$
$$\frac{\theta_4}{2} = 210^\circ$$

However, the multiple 2 takes care of this uncertainty:

$$\theta_4 = 60^\circ \text{ or}$$
$$\theta_4 = 420^\circ = 60^\circ$$

Now, for next time consider the following: Do the solutions for θ_4 always exist? What if $E^2 + F^2 - G^2 = 0$? What if $E^2 + F^2 - G^2 < 0$? Stay tuned . . .

Grashof's Law

Grashof was a German Engineer in the late 1800s. **Grashof's Law** is used to determine the relative rotatability of the input and output links in a 4-bar mechanism:

Crank - full rotation, no limits

Rocker - not full rotation, rotates back-and-forth between limits

Mechanism types (input / output links):

Identify longest, shortest, intermediate 2 links: *L, S, P, Q*

1) If $L + S < P + Q$ Then we call this a *Grashof Mechanism* and there are four different mechanisms and rotation conditions:

Diagrams:

a)

b)

c)

2) If $L + S > P + Q$ Then we call this a *Non-Grashof Mechanism* and the are four different mechanism inversions yield only one rotation condition:

3) If $L + S = P + Q$ Then we call this a *Special Grashof Mechanism* and the four different mechanism inversions yield the identical rotation conditions from 1) *Grashof Mechanism*. However, there is the additional interesting and troublesome feature that the mechanism may jump branches! Centerlines of links can become collinear.

Examples

1) $L = 10$, $S = 4$, $P = 8$, $Q = 7$ – demonstrate the 4 possibilities

2) $L = 10$, $S = 6$, $P = 8$, $Q = 7$ – all Double Rockers

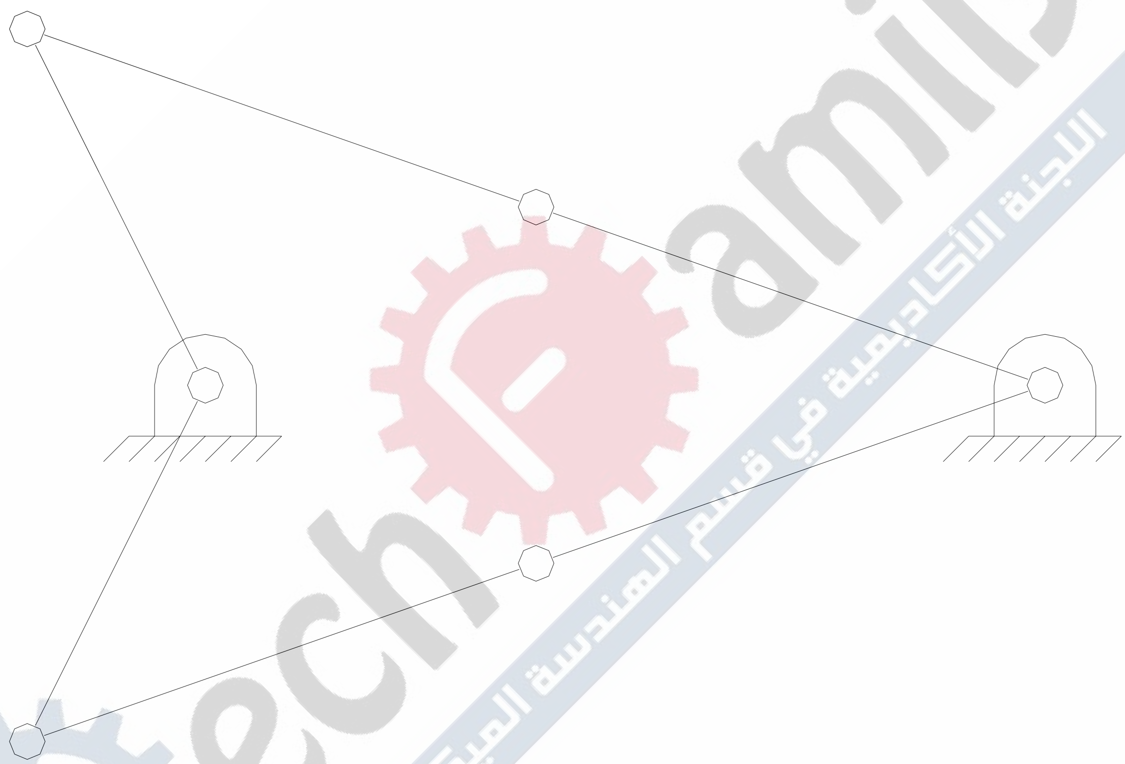
3) $L = 10$, $S = 5$, $P = 8$, $Q = 7$ – demonstrate branch jumping

Another interesting example: $L = P = 10$, $S = Q = 4$
parallel, locomotive linkage – subject to branch jumping unless constrained. Also, very easy analysis:

$$\theta_2 = \theta_4 = \mu \qquad \theta_3 = 0 \text{ for all motion!}$$

4-Bar Joint Limits

If Grashof's Law predicts the input link is a rocker, there will be rotation limits on the input link. These joint limits occur when links 3 and 4 are aligned. As shown in the figure, there will be two joint limits, symmetric about the ground link.



To calculate the joint limits, we use the law of cosines:

$$(r_3 + r_4)^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta_{2L}$$

$$\theta_{2L} = \pm\cos^{-1}\left[\frac{r_1^2 + r_2^2 - (r_3 + r_4)^2}{2r_1 r_2}\right]$$

$\pm$ by symmetry about r_1

Example 1: Given $r_1 = 10, r_2 = 6, r_3 = 8, r_4 = 7$

$$L + S > P + Q \, (10 + 6 > 8 + 7)$$

so we predict only double rockers from this mechanism.

$$\theta_{2L} = \pm\cos^{-1}\left[\frac{10^2 + 6^2 - (8+7)^2}{2(10)(6)}\right] = \pm\cos^{-1}[-0.742] = \pm 137.9^\circ$$

Example 2: Given $r_1 = 10, r_2 = 4, r_3 = 8, r_4 = 7$

$$L + S < P + Q \, (10 + 4 < 8 + 7),$$

so we predict this mechanism is a crank-rocker. Therefore, there are no joint limits!

$$\theta_{2L} = \pm\cos^{-1}\left[\frac{10^2 + 4^2 - (8+7)^2}{2(10)(4)}\right] = \pm\cos^{-1}[-1.3625]$$

which is undefined!

Caution: the figure on the previous page does not apply in all joint limit cases. For certain mechanisms, the limiting conditions occur when links 3 and 4 fold upon each other instead of stretching straight out. The previous method can also be used to find angular limits on link 4 when it is a rocker; here links 2 and 3 either stretch out in a line or fold upon each other.

Example 3: (Term Example Four-bar)

Given $r_1 = 11.18, r_2 = 3, r_3 = 8, r_4 = 7$ (in) and $\theta_1 = 10.3^\circ$, limits are:

$\theta_{4L} = 120.1^\circ$ (links 2 and 3 stretched in a line)

$\theta_{4L} = 172.5^\circ$ (links 2 and 3 folded upon each other in a line)

There are no limits on θ_2 since it is a crank.

Slider-Crank Mechanism Position Analysis

Converts linear motion to rotary or rotary motion to linear via connecting rod. *Internal Combustion Engine* – explosion drives piston (input), output is rotation of drive shaft. *Air Compressor* – electric motor drives crank (input), piston (output) compresses air. Two dead points where piston is at limits. Use flywheel on crank to avoid locking. Unlike the four-bar mechanism, the four kinematic inversions of the slider-crank mechanism yield radically different types of motion. In class we will solve the *Air Compressor* case where the crank is the input and the slider is the output.

Step 1. Draw the **Kinematic Diagram:**

r_2 – input link length θ_2 – input angle
r_3 – coupler link length θ_3 – coupler angle
h – slider offset x – output displacement

Link 1 is the fixed ground link. All angles measured in right-hand sense from horizontal to link. x is measured horizontally from the origin to the slider/coupler revolute joint location.

Step 2. **State the problem:**

Step 3. Draw the **Vector Diagram**. Define all angles in positive sense, measured from the right horizontal to the link vector (tail-to-head).

Vector Diagram:

Step 4. Derive the **Vector-Loop-Closure Equation**. Start at one point, add vectors tail-to-head until reach a second point. Write equation by starting and ending at same points, but choosing a different path.

<u>Step 5.</u> Write <u>***XY* Components**</u> for Vector-Loop-Closure Equation. Break one vector equation into its two scalar components (*X* and *Y*):

<u>Step 6.</u> <u>**Solve for the Unknowns**</u> from the *XY* Equations. Two coupled nonlinear equations in the two unknowns x, θ_3. We could isolate on unknown, square & add, and solve as in the four-bar approach. However, notice that the two *XY* equations are coupled only in θ_3 but not in x. There a simpler method - solve θ_3 using the *Y* equation only and then solve x from the *X* equation:

What about trigonometric uncertainty? The inverse sine function is double-valued and so there are two valid solution branches. Graphically demonstrate the two branches.

Full-rotation condition

For solution to exist for entire motion range (r_2 is a crank), absolute value of the inverse sine argument must be less than or equal 1:

$$\left|\frac{h - r_2 s_2}{r_3}\right| \le 1 \qquad r_3 \ge h - r_2 s_2$$

which must hold for all motion. The worst case is $\theta_2 = -90^\circ$, which yields

$$r_3 \ge h + r_2$$

This condition was derived assuming positive h; allowing negative h:

$$r_3 \ge |h| + r_2 .$$

Slider-Crank Mechanism Position Analysis: Term Example 2

Given:

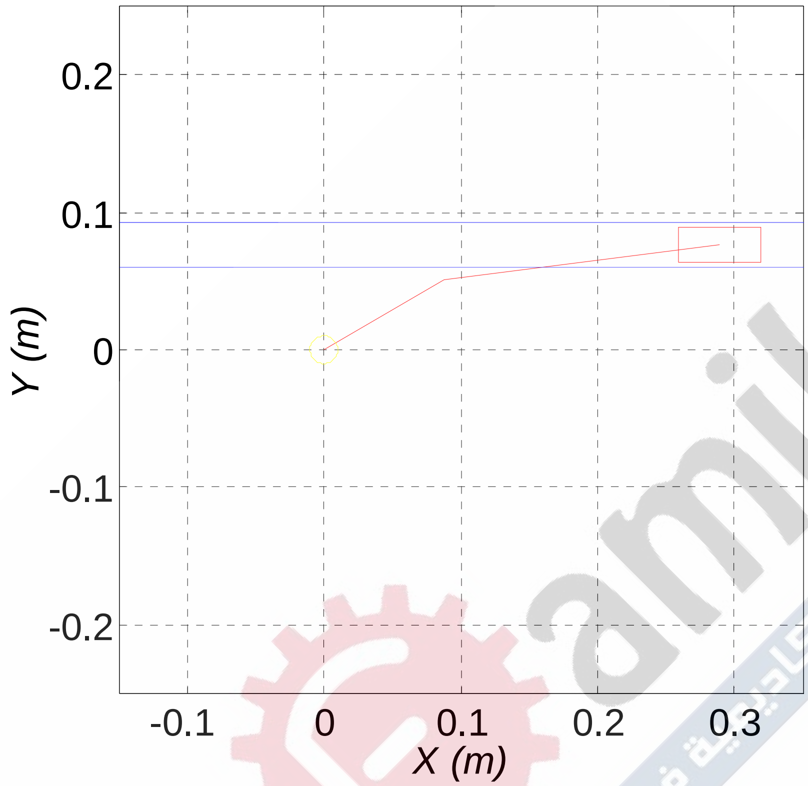
$$r_2 = 4 \qquad\qquad r_2 = 0.102$$
$$r_3 = 8 \ \ in \qquad\qquad r_3 = 0.203 \ \ m$$
$$h = 3 \qquad\qquad h = 0.076$$

Snapshot Analysis (one input angle)

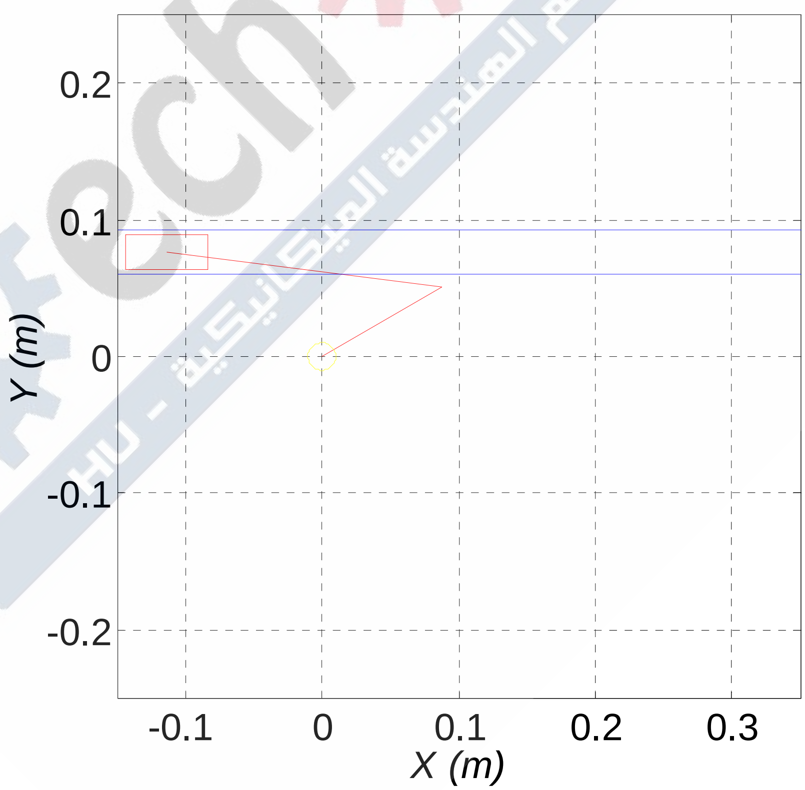
Given this mechanism and $\theta_2 = 30^\circ$, calculate x and θ_3 for both branches. Results:

Branch	x (m)	θ_3
Open	0.290	7.2°
Crossed	-0.114	172.8°

These two branch solutions are demonstrated in the figures on the following page. We use the SI system (m).



Slider-Crank Example Snapshot, Open Branch



Slider-Crank Example Snapshot, Crossed Branch

Graphical Solution: The Slider-Crank position analysis may be solved *graphically*, merely by drawing the mechanism and determining the mechanism closure. This is an excellent method to validate your computer results at a given snapshot.

- Place the grounded revolute for the crank at the origin.
- Draw the line of the slider, offset vertically from the origin by h.
- Draw the given input link 2 length at the given angle (point A).
- Draw a circle of radius r_3, centered at point A.
- This circle intersects the slider line in general in two places.
- Connect the two branches and measure the unknown values.

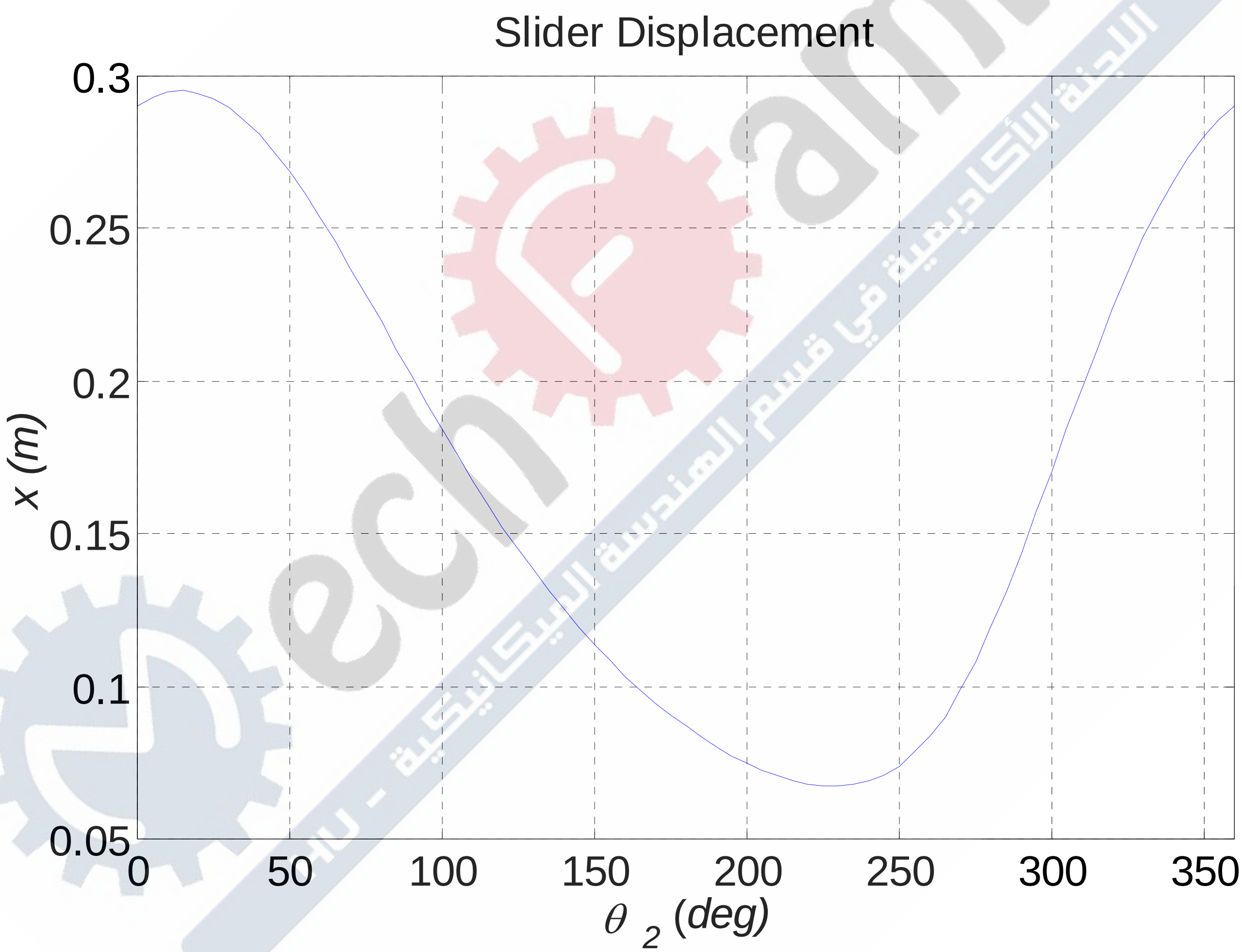
Graphical Solution Figure:

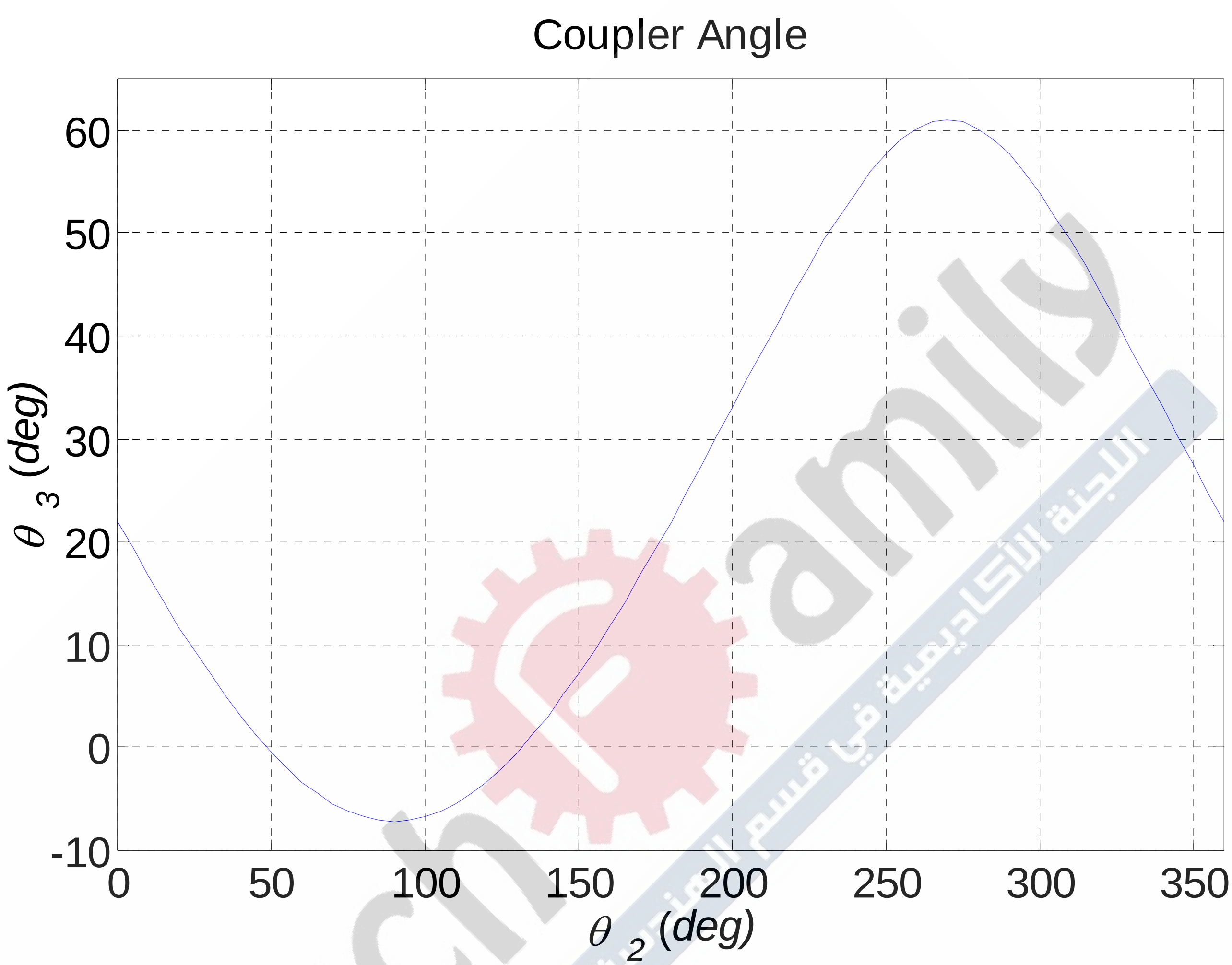
Slider Limits

The crank will rotate fully if the previously-derived condition is met. The slider reaches its maximum displacement when links 2 and 3 are aligned straight out and its maximum displacement when link 2 if folded onto link 3. We can draw two right triangles representing these conditions and easily calculate the x limits to be $0.0671 \le x \le 0.2951$, as seen in the full motion x plot, next page.

Full-Range-Of-Motion (F.R.O.M.) Analysis

A more meaningful result from position analysis is to report the position analysis unknowns for the entire range of mechanism motion. The first plot gives x (m), for all $0^\circ \le \theta_2 \le 360^\circ$, for Term Example 2, right branch only. The second plot gives θ_3 (deg), for all $0^\circ \le \theta_2 \le 360^\circ$, for the right branch only.





Coupler Angle
60
50
40
30
20
10
0
-10
0
50
100
150
200
250
300
350
θ 3 (deg)
θ 2 (deg)

Velocity Analysis Introduction

Velocity analysis is important for kinematic motion analysis. Some tasks have timing, rates. Position analysis must be completed first. Velocity analysis is also required for dynamics: position, ***velocity***, acceleration, dynamics, forces, machine design. Velocity analysis is solution of coupled *linear* equations. Velocity is the first time derivative of the position. Vector quantity:

Magnitude of velocity is speed; direction also crucial. Analytical velocity analysis: write position vectors, take first time derivatives, solve for unknowns. Units (translational and rotational):

Basic Velocity Derivation Figure:

Most general planar case: Translating and rotating rigid rod with a slider on it. Find the total velocity of point *P* on the slider. Express the position vector in Cartesian coordinates:

$\underline{P}_P = \underline{P}_O + \underline{L} =$

The angle is changing with angular velocity:

Only the planar case is this simple; the spatial rotation case is more complicated. The length of the rod is changing with sliding velocity:

Product and Chain Rules of Differentiation

We'll need to use the product and chain rules over and over in velocity and acceleration analysis derivations.

Product rule:

$$\frac{d}{dt}(xy) = \frac{dx}{dt}y + x\frac{dy}{dt}$$ x, y both functions of time.

Chain rule:

$$\frac{d}{dt}(f(x(t))) = \frac{df}{dx}\frac{dx}{dt}$$ f is a function of x, which is a function of t.

Example:

$$\frac{d}{dt}(L\cos\theta) = ?$$

Back to basic velocity derivation

<u>First time derivative of position vector:</u>

$$\underline{V}_P = \frac{d\underline{P}_P}{dt} =$$

We have just derived the **Three-Part Velocity Equation:**

$$\underline{V}_P = \underline{V}_O + \underline{V} + \underline{\omega} \times \underline{L}$$

The terms for the **Three-Part Velocity Equation** can be expressed in various ways, summarized below:

Vector	$\underline{V}_O$	$\underline{V}$	$\underline{\omega} \times \underline{L}$
Name	Point O Velocity	Sliding Velocity	Tangential Velocity
***XY* Components**			
Magnitude / Direction			

Three-Part Velocity Equation Example:

Given (instantaneously) $L = 2\ m$, $\theta = 30^\circ$, $\omega = 1\ rad/s$, $|\underline{V}| = \dot{L} = 3\ m/s$ (outward), $\underline{V}_O = \{3 \quad 2\}^T\ m/s$, calculate $\underline{V}_P$.

$$\underline{V}_P = \begin{Bmatrix} V_{OX} + V\cos\theta - L\omega\sin\theta \\ V_{OY} + V\sin\theta + L\omega\cos\theta \end{Bmatrix} = \begin{Bmatrix} 3 + 3\cos 30^\circ - 2(1)\sin 30^\circ \\ 2 + 3\sin 30^\circ + 2(1)\cos 30^\circ \end{Bmatrix}$$

$$\underline{V}_P = \begin{Bmatrix} 3 + 2.598 - 1 \\ 2 + 1.5 + 1.732 \end{Bmatrix} = \begin{Bmatrix} 4.598 \\ 5.232 \end{Bmatrix} \frac{m}{s}$$

or, $$\underline{V}_P = 6.965 @ 48.7^\circ\ \ m/s$$

Show magnitude and direction of each velocity component:

Vector	$\underline{V}_O$	$\underline{V}$	$\underline{\omega} \times \underline{L}$
Name	Point O Velocity	Sliding Velocity	Tangential Velocity
***XY* Components**			
Magnitude / Direction			

Four-Bar Mechanism Velocity Analysis

Velocity Analysis: determination of angular and linear velocities of links in a mechanism. Required for complete motion analysis. Also required for further analysis: acceleration, dynamics, forces, machine design. Linear equations result from first time differentiation of position equations. Unique solution for each mechanism branch. Position analysis must be complete first. 1-dof mechanism, so one velocity input must be given.

Generic Mechanism Velocity Analysis Statement: Given the mechanism, complete position analysis, and one dof of velocity input, calculate the velocity unknowns.

Four-bar Mechanism Velocity Analysis

Step 1. **Position Analysis** must first be complete.

Step 2. Draw the **Velocity Diagram:**

where $\underline{\omega}_i$, $i = 2,3,4$, is the absolute angular velocity of link i. $\underline{\omega}_1 = 0$ since the ground link is fixed.

Step 3. **State the problem:**

Step 4. **Derive the velocity equations**. Take the first time derivative of the vector loop closure equations from position analysis, in *XY* component form.

Four-bar mechanism position equations:

$$\underline{r}_2 + \underline{r}_3 = \underline{r}_1 + \underline{r}_4$$

$$r_2 c_2 + r_3 c_3 = r_1 c_1 + r_4 c_4$$
$$r_2 s_2 + r_3 s_3 = r_1 s_1 + r_4 s_4$$

First time derivative for velocity equations: (use chain rule several times) Chain rule:

$$\frac{d}{dt}(\cos\theta_i) = \frac{d\cos\theta_i}{d\theta_i}\frac{d\theta_i}{dt} = -\sin\theta_i\dot{\theta}_i = -\sin\theta_i\omega_i$$

$$\frac{d}{dt}(\sin\theta_i) = \frac{d\sin\theta_i}{d\theta_i}\frac{d\theta_i}{dt} = \cos\theta_i\dot{\theta}_i = \cos\theta_i\omega_i$$

Don't have to use product rule because $\dot{r}_i = 0$ (rigid links).

The first time derivative of the position equations is:

Gathering unknowns on the LHS:

Substituting simpler terms:

Written in matrix form:

<u>Step 5.</u> **<u>Solve the velocity equations</u>** for the unknowns ω_3, ω_4.

Algebra solution:

Alternate **matrix solution** (yields same solution):

Four-Bar mechanism singularity condition:

When does the solution fail? This is a mechanism singularity, when the determinant of the coefficient matrix goes to zero. The result is dividing by zero, for infinite answers ω_3, ω_4. Let's see what this means physically.

Physically, this happens when links 3 and 4 are straight out or folded on top of each other (what does this correspond to?):

The basic four-bar mechanism velocity analysis problem is now solved. Now that we know the angular unknowns, we can find the **translational velocity of any point** on the mechanism, e.g. coupler point C:

Four-bar mechanism velocity example:

Given r_1 = 0.284, r_2 = 0.076, r_3 = 0.203, r_4 = 0.178, $R_{C/A}$ = 0.127 m, and $\theta_1 = 10.3^\circ$, $\theta_2 = 30^\circ$, $\theta_3 = 53.8^\circ$, $\theta_4 = 121.7^\circ$, $\delta_3 = 36.9^\circ$. This is the open branch of the four-bar mechanism position example (Term Example 1).

Snapshot Analysis (one input angle)

Given this mechanism position analysis plus $\omega_2 = \pi \ rad/s$ (+, so ccw), calculate ω_3, ω_4, and $\underline{V}_C$ for this instant (snapshot).

$$\begin{bmatrix} 0.164 & -0.151 \\ -0.120 & -0.093 \end{bmatrix} \begin{Bmatrix} \omega_3 \\ \omega_4 \end{Bmatrix} = \begin{Bmatrix} -0.120 \\ 0.207 \end{Bmatrix}$$

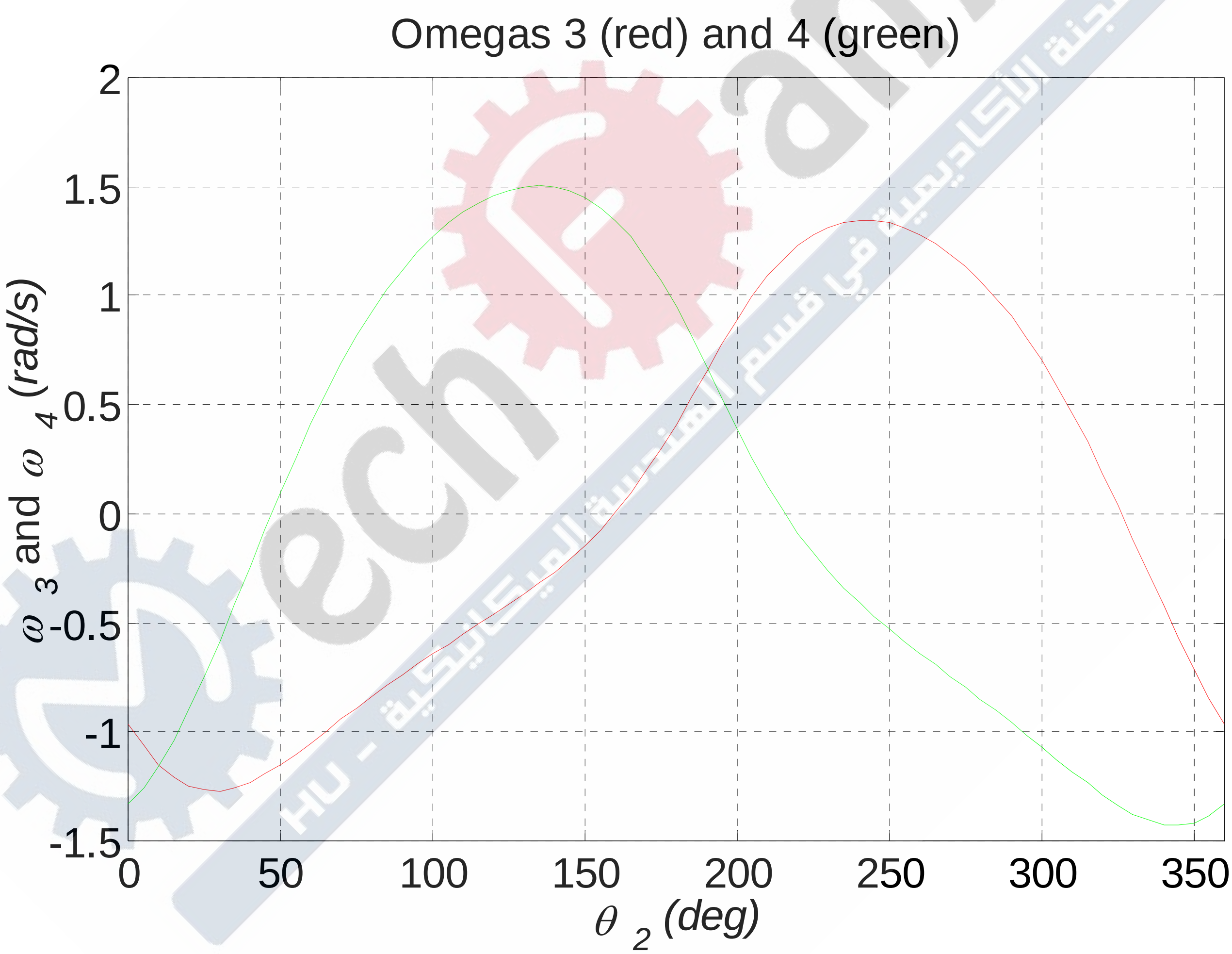
$$\begin{Bmatrix} \omega_3 \\ \omega_4 \end{Bmatrix} = \begin{Bmatrix} -1.271 \\ -0.587 \end{Bmatrix}$$

Both are negative, so cw direction. These results are the absolute angular velocities of links 3 and 4 with respect to the ground link.

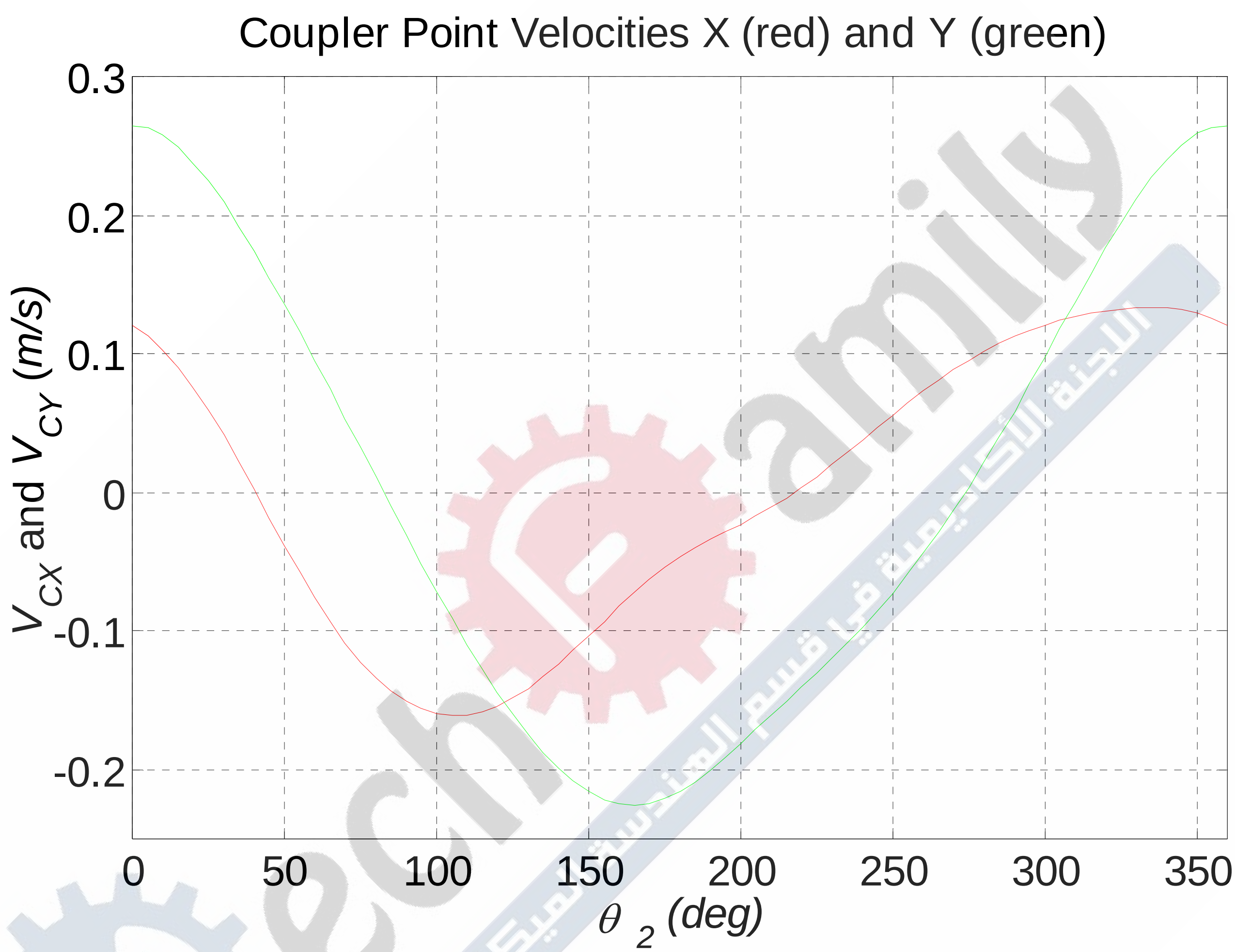
Coupler point translational velocity: $\underline{V}_C = \begin{Bmatrix} 0.042 \\ 0.209 \end{Bmatrix}$ (m/s)

Full-Range-Of-Motion (F.R.O.M.) Analysis

A more meaningful result from velocity analysis is to report the velocity analysis unknowns for the entire range of mechanism motion. The plot below gives ω_3 (red) and ω_4 (green), (rad/s), for all $0^\circ \le \theta_2 \le 360^\circ$, for Term Example 1, open branch only. Since ω_2 is constant, we can plot the velocity results vs. θ_2 (since it is related to time t via $\theta_2 = \omega_2 t$).



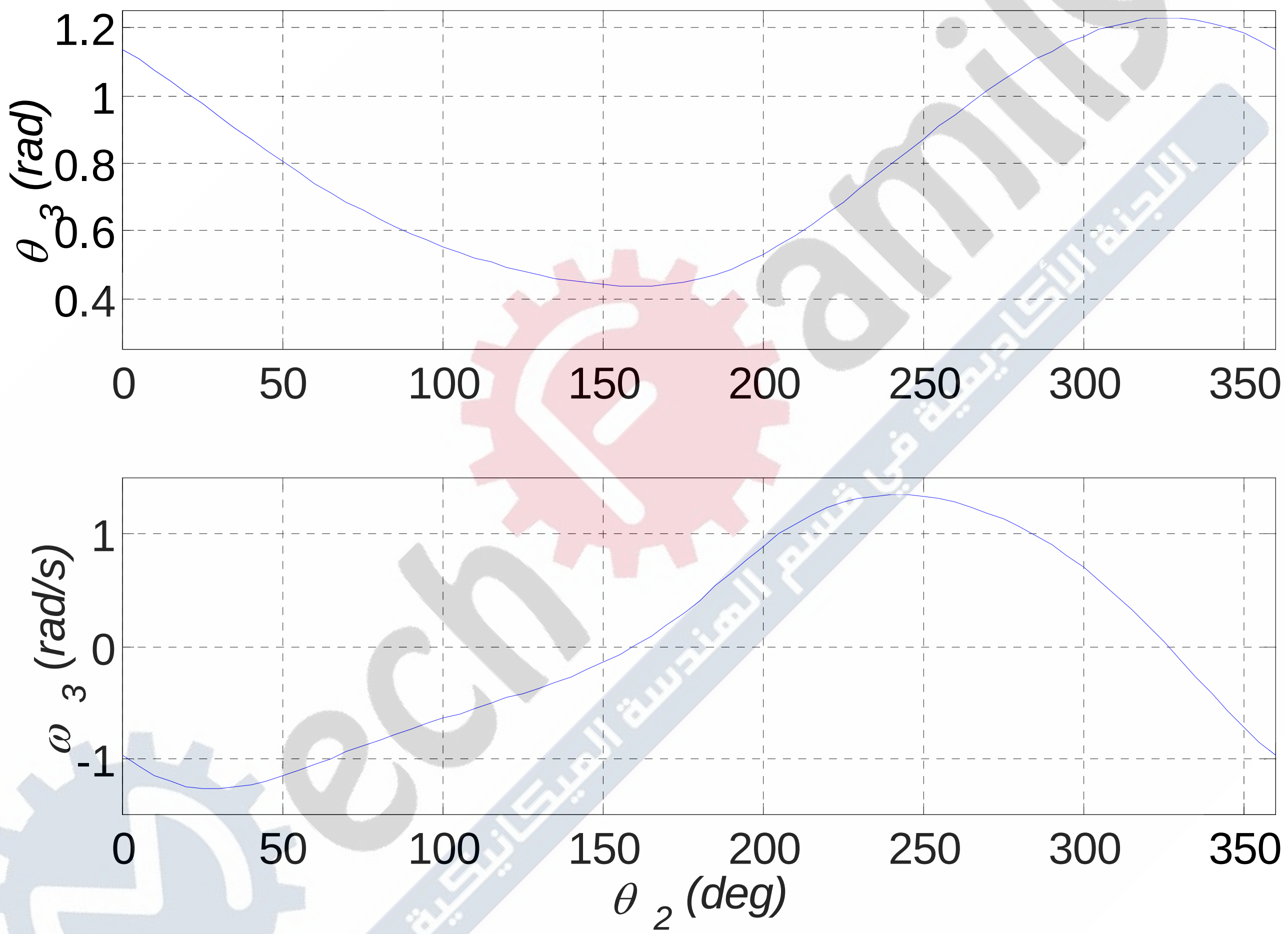
The plot below gives the translational coupler point velocity for all $0^\circ \le \theta_2 \le 360^\circ$, for Term Example 1, open branch only.



Derivative/Integral Relationships

When one variable is the derivative of another, what are the relationships? For example:

$$\omega_3 = \frac{d\theta_3}{dt} \qquad\qquad \theta_3 = \theta_{30} + \int \omega_3 dt$$



The value of ω_3 at any point is the slope of the θ_3 curve at that point. The value of θ_3 at any point is the integral of the ω_3 curve up to that point (the value of θ_3 at any point is the area under the ω_3 curve up to that point).

This graph is vs. θ_2, but the same type of relationships hold as for time t since ω_2 is constant. This is the Term Example 1 result, but we changed θ_3 from *deg* to *rad* for better comparison.

Slider-Crank Mechanism Velocity Analysis

Again, we will solve the *Air Compressor* case where the crank is the input and the slider is the output. The *Internal Combustion Engine* case (slider input/crank output) is equally interesting.

Step 1. **Position Analysis** must first be complete.

Step 2. Draw the **Velocity Diagram:**

where $\underline{\omega}_i$, $i = 2,3$ is the absolute angular velocity of link i. $\dot{x}$ is the variable slider velocity. $\underline{\omega}_4 = 0$ since the slider cannot rotate.

Step 3. **State the problem:**

Step 4. **Derive the velocity equations**. Take the first time derivative of the vector loop closure equations from position analysis, in *XY* component form.

Slider-crank mechanism position equations:

$$\underline{r}_2 + \underline{r}_3 = \underline{x} + \underline{h}$$

$$r_2 c_2 + r_3 c_3 = x$$
$$r_2 s_2 + r_3 s_3 = h$$

First time derivative for velocity equations:

Gathering unknowns on the LHS:

Written in matrix form:

Step 5. **Solve the velocity equations** for the unknowns $\omega_3, \dot{x}$.

Actually, these equations are decoupled so we don't need a matrix solution. First, solve ω_3 from Y equation:

Then solve $\dot{x}$ from the X equation using the ω_3 result:

The alternate matrix solution:

will yield identical results.

Slider-crank mechanism singularity condition:

When does the solution fail? This is a slider-crank mechanism singularity, when the determinant of the coefficient matrix goes to zero. The result is dividing by zero, resulting in infinite answers $\omega_3, \dot{x}$.

$$|A| = r_3 c_3 = 0$$

$$|A| = 0 \text{ when } \cos\theta_3 = 0\text{, or} \qquad \theta_3 = 90^\circ, 270^\circ, \cdots$$

Physically, this happens when link 3 is straight up or down ($\theta_3 = \pm 90^\circ$). Doesn't happen for nominal full-rotation slider-crank mechanisms, even with offsets.

Of course r_3 cannot go to zero, otherwise we have a degenerate slider-crank mechanism.

Slider-crank mechanism velocity example:

Given r_2 = 0.102, r_3 = 0.203, h = 0.076 m, and $\theta_2 = 30^\circ$, $\theta_3 = 7.2^\circ$, x = 0.290 m. This is the right branch of the slider-crank position example (Term Example 2).

Snapshot Analysis (one input angle)

Given this mechanism position analysis plus $\omega_2 = \pi/2$ rad/s (+, so ccw), calculate $\dot{x}, \omega_3$ for this instant (snapshot).

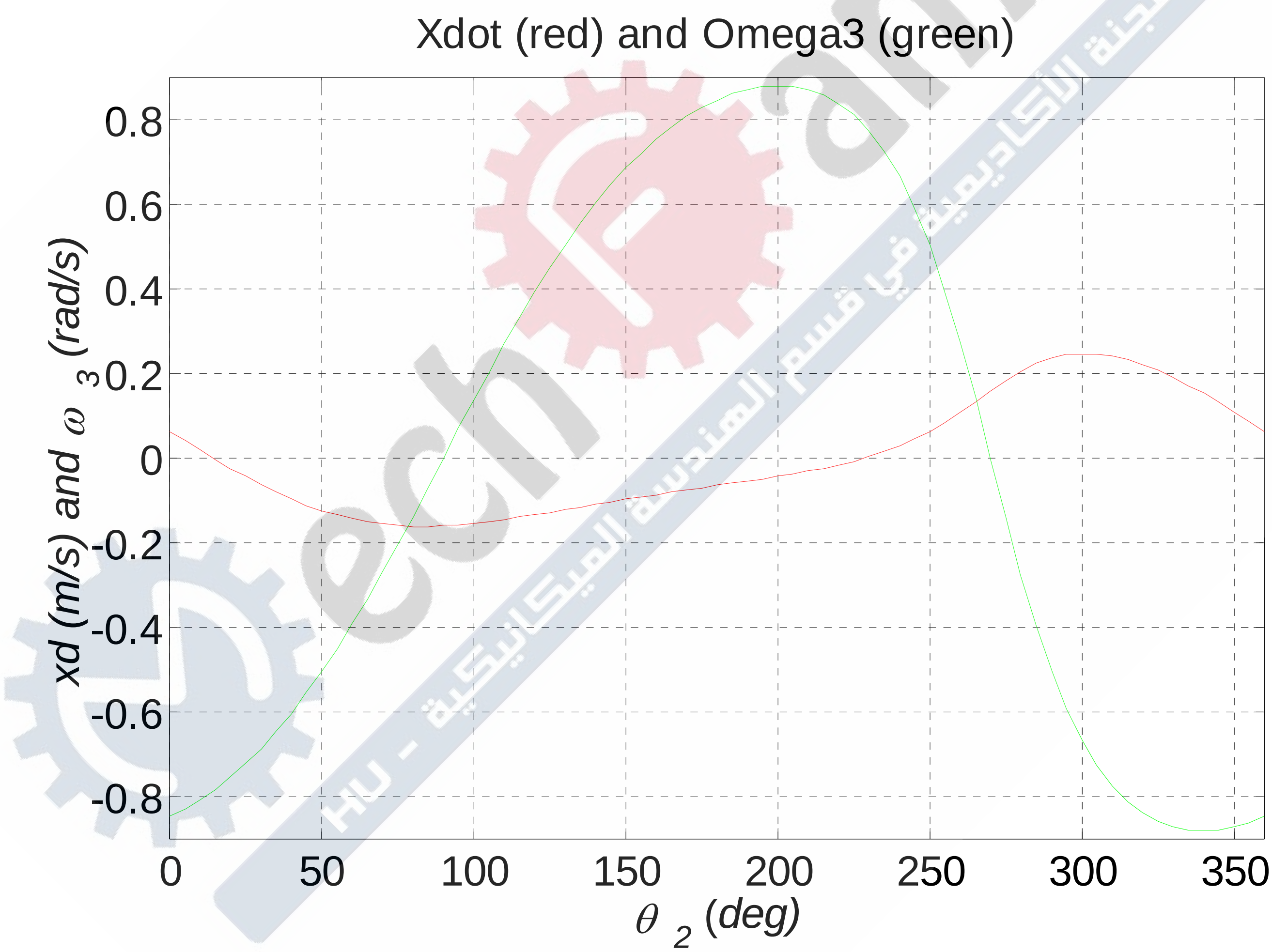
$$\begin{bmatrix} 1 & 0.025 \\ 0 & -0.202 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} -0.080 \\ 0.138 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{x} \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} -0.062 \\ -0.686 \end{Bmatrix}$$

Both are negative, so the slider is currently traveling to the left and the coupler link is currently rotating in the cw direction. These results are the absolute linear and angular velocities of links 4 and 3 with respect to the fixed ground link.

Full-Range-Of-Motion (F.R.O.M.) Analysis

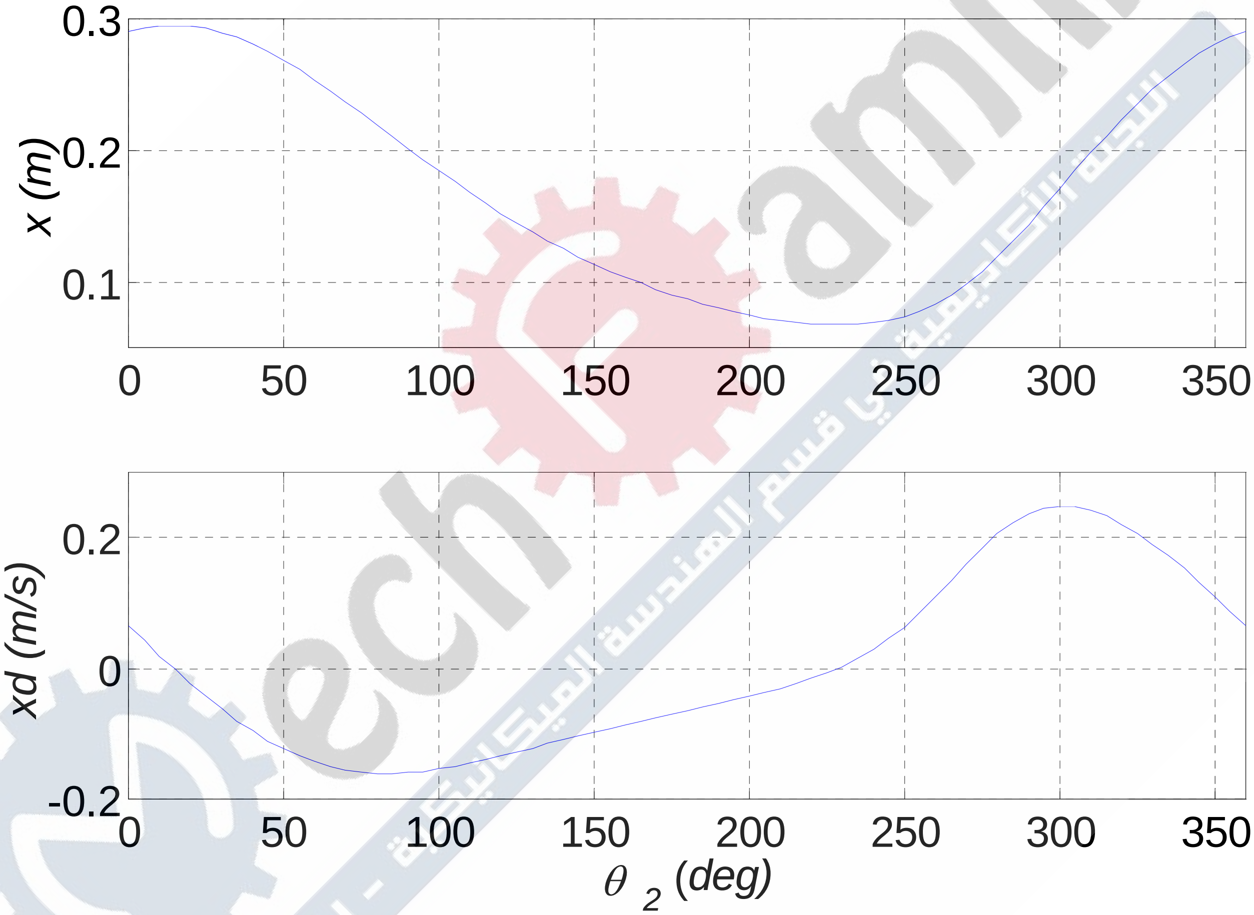
A more meaningful result from velocity analysis is to report the velocity analysis unknowns for the entire range of mechanism motion. The plot below gives $\dot{x}$ (red, m/s) and ω_3 (green, rad/s), for all $0^\circ \le \theta_2 \le 360^\circ$, for Term Example 2, right branch only. Since ω_2 is constant, we can plot the velocity results vs. θ_2 (since it is related to time t via $\theta_2 = \omega_2 t$).



Derivative/Integral Relationships

When one variable is the derivative of another, what are the relationships? For example:

$$\dot{x} = \frac{dx}{dt} \qquad\qquad x = x_0 + \int \dot{x}dt$$



The value of $\dot{x}$ at any point is the slope of the x curve at that point. The value of x at any point is the integral of the $\dot{x}$ curve up to that point (the value of x at any point is the area under the $\dot{x}$ curve up to that point).

This graph is vs. θ_2, but the same type of relationships hold as for time t since ω_2 is constant. This is the Term Example 2 result.

Acceleration Analysis Introduction

Acceleration analysis is required for dynamics: position, velocity, ***acceleration***, dynamics, forces, machine design. Important for kinematic motion analysis. Position and velocity analyses must be completed first. Acceleration analysis is solution of linear equations. Acceleration is the first time derivative of the velocity and second time derivative of the position. Vector quantity:

Analytical acceleration analysis: write position vectors, take first two time derivatives, solve for unknowns. Units (translational and rotational):

Basic Acceleration Derivation Figure:

Rotating rigid rod with a slider on it. Find the total acceleration of point *P* on the slider.

Recall the 2-part position and 3-part velocity results:

$$\underline{P}_P = \underline{P}_O + \underline{L} = \begin{Bmatrix} P_{OX} + L\cos\theta \\ P_{OY} + L\sin\theta \end{Bmatrix}$$

$$\underline{V}_P = \underline{V}_O + \underline{V} + \underline{\omega} \times \underline{L} = \begin{Bmatrix} V_{OX} + V\cos\theta - L\omega\sin\theta \\ V_{OY} + V\sin\theta + L\omega\cos\theta \end{Bmatrix}$$

The angle is changing with angular velocity and acceleration:

Only planar case is this simple; the spatial rotation case is more complicated. The length of the rod is changing with sliding velocity and acceleration:

Product and Chain Rules of Differentiation

Again, we'll need to use the product and chain rules over and over in acceleration analysis derivations.

Product rule:

$$\frac{d}{dt}(xy) = \frac{dx}{dt}y + x\frac{dy}{dt}$$ x, y both functions of time.

Chain rule:

$$\frac{d}{dt}(f(x(t))) = \frac{df}{dx}\frac{dx}{dt}$$ f is a function of x, which is a function of t.

Example:

$$\frac{d^2}{dt^2}(L\cos\theta) = ?$$

Back to basic acceleration derivation

<u>First time derivative of velocity vector (Second time derivative of position vector):</u>

$$\underline{A}_P = \frac{d\underline{V}_P}{dt} = \frac{d^2\underline{P}_P}{dt^2} =$$

We have just derived the **Five-Part Acceleration Equation:**

$$\underline{A}_P = \underline{A}_O + \underline{A} + 2\underline{\omega} \times \underline{V} + \underline{\alpha} \times \underline{L} + \underline{\omega} \times (\underline{\omega} \times \underline{L})$$

These terms can be expressed in various ways, summarized below:

Vector	$\underline{A}_O$	$\underline{A}$	$2\underline{\omega} \times \underline{V}$	$\underline{\alpha} \times \underline{L}$	$\underline{\omega} \times (\underline{\omega} \times \underline{L})$
Name	Point O Acceleration	Sliding Acceleration	Coriolis Acceleration	Tangential Acceleration	Centripetal Acceleration
***XY* Components**					
Magnitude / Direction					

Five-Part Acceleration Equation Example:
Continuation of 3-part velocity example.

Given (instantaneously) L=2 m, $\theta = 30^\circ$, $\omega = 1\ rad/s$, $\alpha = 2\ rad/s^2$, $|\underline{V}| = \dot{L} = 3\ m/s$ (outward), $\underline{V}_O = \{3 \quad 2\}^T$, $|\underline{A}| = \ddot{L} = 4\ m/s^2$ (outward), $\underline{A}_O = \{1 \quad 2\}^T$, calculate $\underline{A}_P$.

$$\underline{A}_P = \begin{Bmatrix} A_{OX} + A\cos\theta - 2V\omega\sin\theta - L\alpha\sin\theta - L\omega^2\cos\theta \\ A_{OY} + A\sin\theta + 2V\omega\cos\theta + L\alpha\cos\theta - L\omega^2\sin\theta \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 + 3.464 - 3 - 2 - 1.732 \\ 2 + 2 + 5.196 + 3.464 - 1 \end{Bmatrix} = \begin{Bmatrix} -2.268 \\ 11.660 \end{Bmatrix} \frac{m}{s^2}$$

or, $\underline{A}_P = 11.879 @ 101.0^\circ\ m/s^2$

Show magnitude and direction of each Acceleration component:

Vector	$\underline{A}_O$	$\underline{A}$	$2\underline{\omega} \times \underline{V}$	$\underline{\alpha} \times \underline{L}$	$\underline{\omega} \times (\underline{\omega} \times \underline{L})$
Name	Point O Acceleration	Sliding Acceleration	Coriolis Acceleration	Tangential Acceleration	Centripetal Acceleration
***XY* Components**					
Magnitude / Direction					

Four-Bar Mechanism Acceleration Analysis

Acceleration Analysis - determination of angular and linear accelerations of links in a mechanism. Required for complete motion analysis. Also required for further analysis: dynamics, forces, machine design. Linear equations result from second time differentiation of position equations. Unique solution for each mechanism branch. Position and velocity analyses must be complete first. 1-dof mechanism, so one acceleration input must be given.

Generic Mechanism Acceleration Analysis Statement: Given the mechanism, complete position and velocity analyses, and one dof of acceleration input, calculate the acceleration unknowns.

Four-bar Mechanism Acceleration Analysis

Step 1. **Position and Velocity Analyses** must first be complete.

Step 2. Draw the **Acceleration Diagram:**

where $\underline{\alpha}_i$, $i = 2,3,4$ is the absolute angular acceleration of link i. $\underline{\alpha}_1 = 0$ since the ground link is fixed.

Step 3. **State the problem:**

Step 4. **Derive the acceleration equations**. Take the first time derivative of the four-bar mechanism velocity equations from velocity analysis, in *XY* component form.

Four-bar mechanism velocity equations:

$$-r_2\omega_2 s_2 - r_3\omega_3 s_3 = -r_4\omega_4 s_4$$
$$r_2\omega_2 c_2 + r_3\omega_3 c_3 = r_4\omega_4 c_4$$

The first time derivative of the velocity equations is:

Gathering unknowns on the LHS:

Substituting simpler terms:

Written in matrix form:

<u>Step 5.</u> **<u>Solve the acceleration equations</u>** for the unknowns α_3, α_4.

Matrix solution (Algebra solution yields the same results):

Four-Bar mechanism singularity condition:

Same coefficient matrix A as velocity case, so singularity condition is identical:

$$\theta_4 - \theta_3 = 0^\circ, 180^\circ, \cdots$$

This condition is the same problem for position, velocity, and acceleration. At this singularity, there is zero transmission angle μ and Link 2 is at a joint limit!

The basic four-bar mechanism acceleration analysis problem is now solved. Now that we know the angular unknowns, we can find the **translational acceleration of any point** on the mechanism, e.g. coupler point C:

Four-bar mechanism acceleration example:

Given $r_1 = 0.284$, $r_2 = 0.076$, $r_3 = 0.203$, $r_4 = 0.178$, $R_{C/A} = 0.127$ m, and $\theta_1 = 10.3^\circ$, $\theta_2 = 30^\circ$, $\theta_3 = 53.8^\circ$, $\theta_4 = 121.7^\circ$, $\delta_3 = 36.9^\circ$; $\omega_2 = \pi$, $\omega_3 = -1.271$, $\omega_4 = -0.587$ rad/s. This is the open branch of the position and velocity example (Term Example 1).

Snapshot Analysis (one input angle)

Given this mechanism position and velocity analysis, plus $\alpha_2 = 0$ rad/s^2, calculate α_3, α_4 for this instant (snapshot).

$$\begin{bmatrix} 0.164 & -0.151 \\ -0.120 & -0.093 \end{bmatrix} \begin{Bmatrix} \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{Bmatrix} -0.877 \\ -0.589 \end{Bmatrix}$$

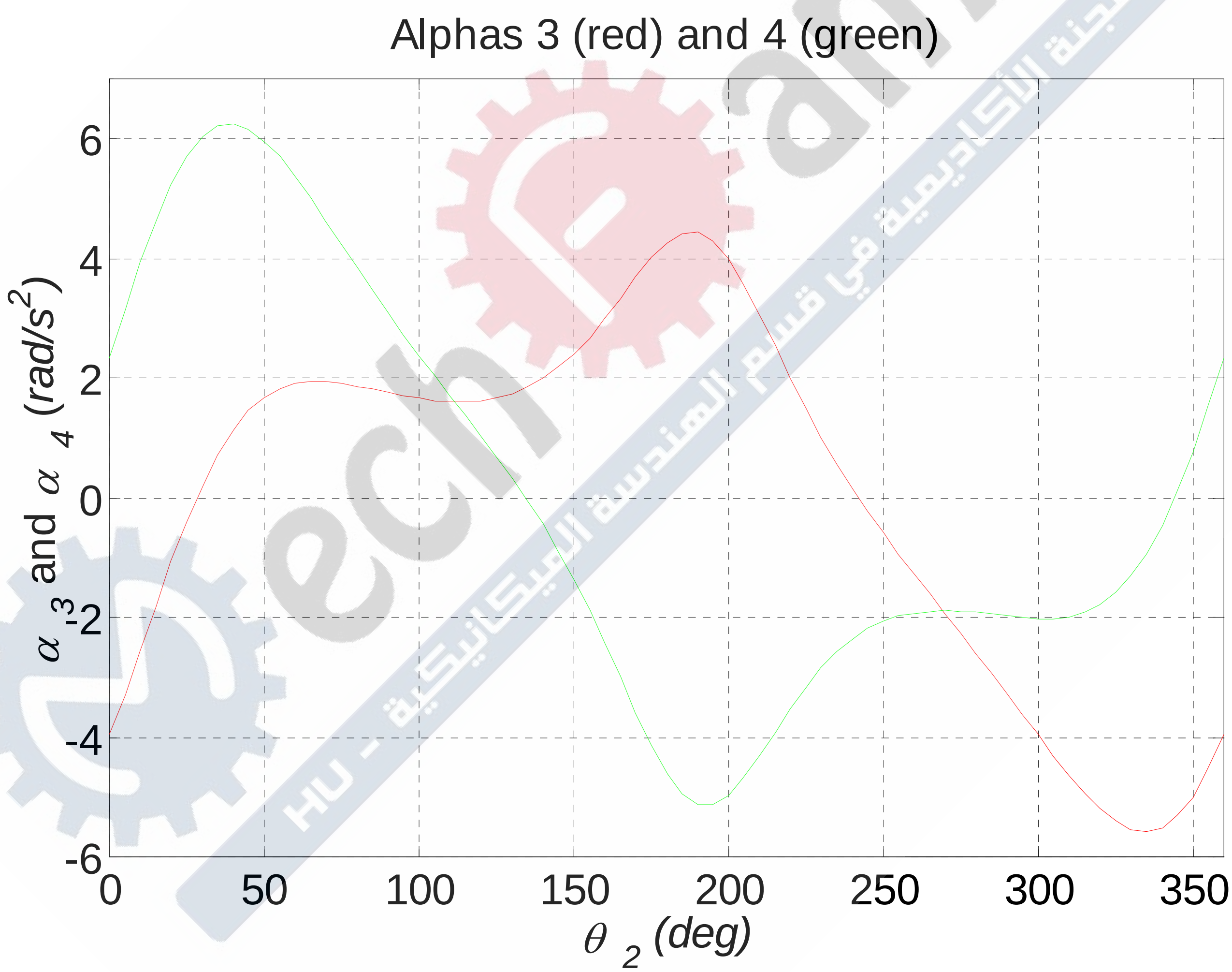
$$\begin{Bmatrix} \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{Bmatrix} 0.213 \\ 6.030 \end{Bmatrix}$$

Both are positive, so *ccw* direction. These results are the absolute angular accelerations of links 3 and 4 with respect to the ground link.

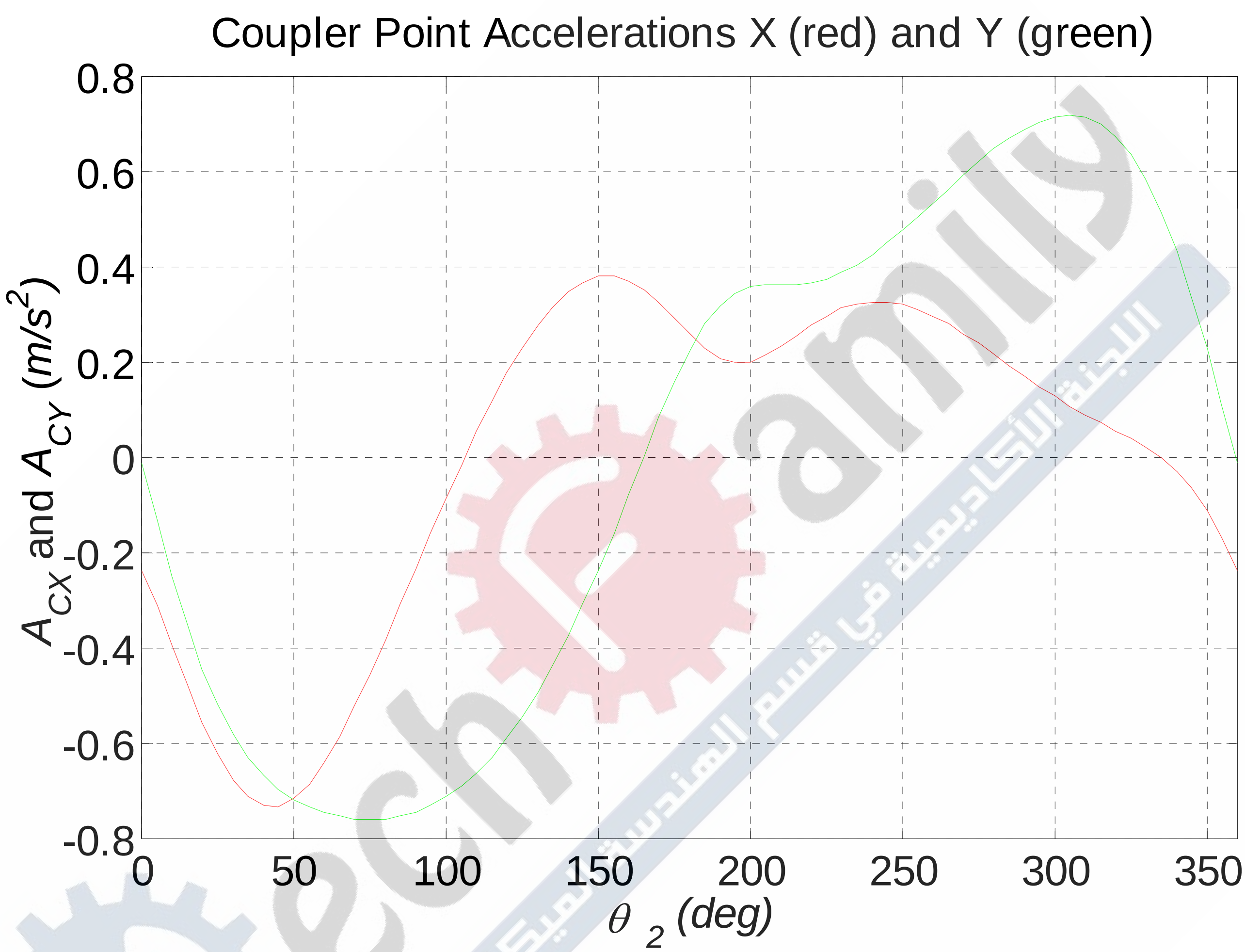
Coupler point translational acceleration: $\underline{A}_C = \begin{Bmatrix} -0.676 \\ -0.582 \end{Bmatrix} m/s^2$

Full-Range-Of-Motion (F.R.O.M.) Analysis

A more meaningful result from acceleration analysis is to report the acceleration analysis unknowns for the entire range of mechanism motion. The plot below gives α_3 (red) and α_4 (green), (rad/s^2), for all $0^\circ \le \theta_2 \le 360^\circ$, for Term Example 1, open branch only. Since ω_2 is constant, we can plot the acceleration results vs. θ_2 (since it is related to time t via $\theta_2 = \omega_2 t$).



The plot below gives the translational coupler point acceleration for all $0^\circ \le \theta_2 \le 360^\circ$, for Term Example 1, open branch only.

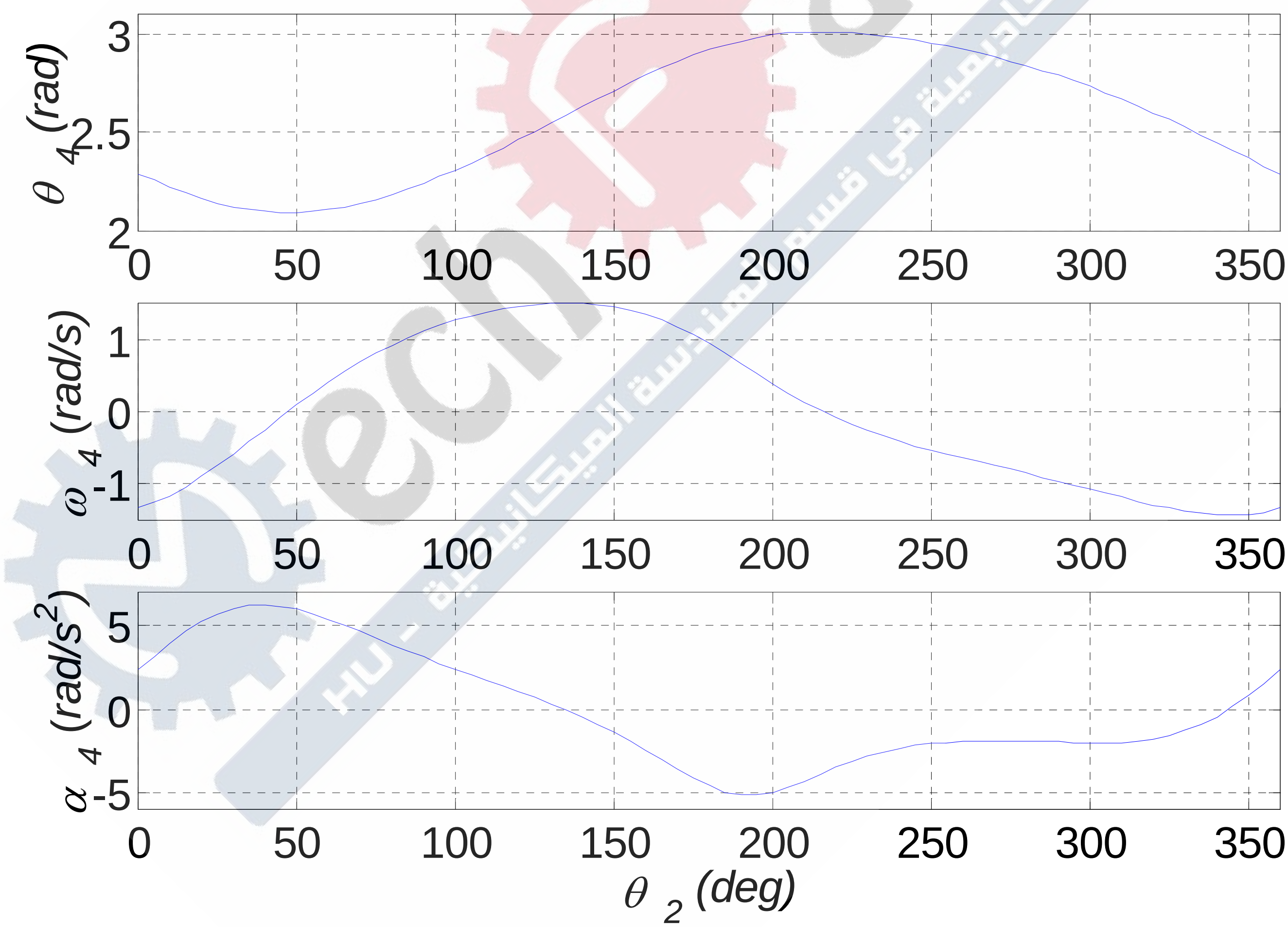


Derivative/Integral Relationships

When one variable is the derivative of another, recall the relationships from calculus. For example:

$$\omega_4 = \frac{d\theta_4}{dt} \qquad \theta_4 = \theta_{40} + \int \omega_4 dt$$

$$\alpha_4 = \frac{d\omega_4}{dt} \qquad \omega_4 = \omega_{40} + \int \alpha_4 dt$$



Slider-Crank Mechanism Acceleration Analysis

Again, we will solve the *Air Compressor* case where the crank is the input and the slider is the output.

Step 1. **Position and Velocity Analyses** must first be complete.

Step 2. Draw the **Acceleration Diagram:**

where $\underline{\alpha}_i; i = 2,3$ is the absolute angular acceleration of link *i*. $\underline{\alpha}_4 = 0$ since the slider cannot rotate.

Step 3. **State the problem:**

Step 4. **Derive the acceleration equations**. Take the first time derivative of the velocity equations from velocity analysis, in *XY* component form.

Slider-crank mechanism velocity equations:

$$-r_2\omega_2 s_2 - r_3\omega_3 s_3 = \dot{x}$$
$$r_2\omega_2 c_2 + r_3\omega_3 c_3 = 0$$

The first time derivative of the velocity equations is:

Gathering unknowns on the LHS:

Written in matrix form:

Step 5. **Solve the acceleration equations** for the unknowns $\alpha_3, \ddot{x}$.

Actually, these equations are decoupled so we don't need a matrix solution. First, solve α_3 from Y equation:

Then solve $\ddot{x}$ from the X equation using the α_3 result:

Slider-crank mechanism singularity condition:

Same coefficient matrix as velocity case, so singularity condition is identical (see the singularity discussion in the slider-crank velocity section).

Slider-crank mechanism acceleration example:

Given r_2 = 0.102, r_3 = 0.203, h = 0.076 m, and $\theta_2 = 30^\circ$, $\theta_3 = 7.2^\circ$, x = 0.290 m; and $\omega_2 = \pi/2$, $\omega_3 = -0.686$ rad/s, $\dot{x} = -0.062$ m/s. This is the right branch of the position and velocity example (Term Example 2).

Snapshot Analysis (one input angle)

Given this mechanism position and velocity analysis plus, $\alpha_2 = 0$ rad/s^2, calculate $\ddot{x}, \alpha_3$ for this instant (snapshot).

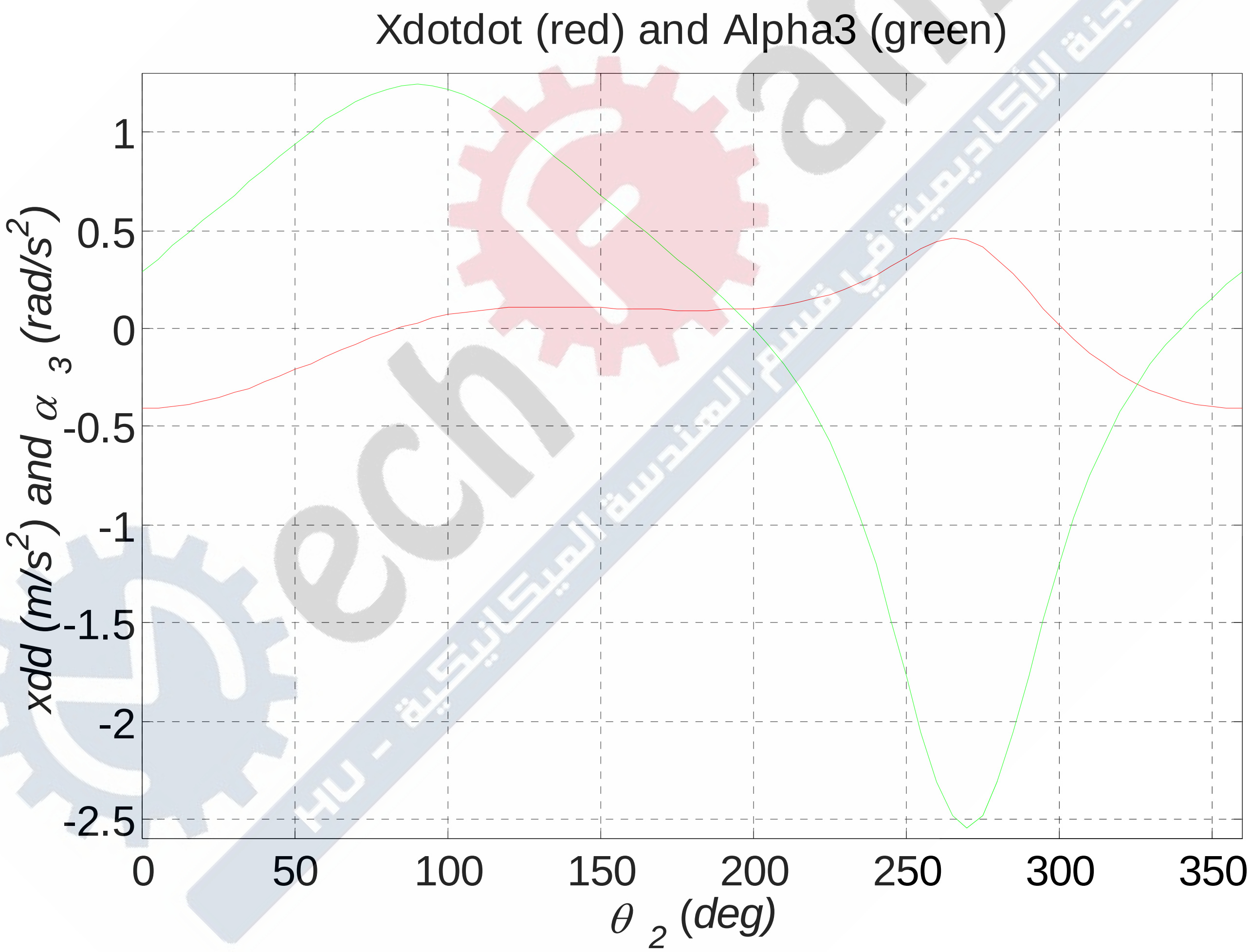
$$\begin{bmatrix} 1 & 0.025 \\ 0 & -0.202 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \alpha_3 \end{Bmatrix} = \begin{Bmatrix} -0.312 \\ -0.137 \end{Bmatrix}$$

$$\begin{Bmatrix} \ddot{x} \\ \alpha_3 \end{Bmatrix} = \begin{Bmatrix} -0.329 \\ 0.681 \end{Bmatrix}$$

These results are the absolute linear and angular accelerations of links 4 and 3 with respect to the fixed ground link.

Full-Range-Of-Motion (F.R.O.M.) Analysis

A more meaningful result from acceleration analysis is to report the acceleration analysis unknowns for the entire range of mechanism motion. The plot below gives $\ddot{x}$ (red, m/s^2) and α_3 (green, rad/s^2), for all $0^\circ \le \theta_2 \le 360^\circ$, for Term Example 2, right branch only. Since ω_2 is constant, we can plot the velocity results vs. θ_2 (since it is related to time t via $\theta_2 = \omega_2 t$).

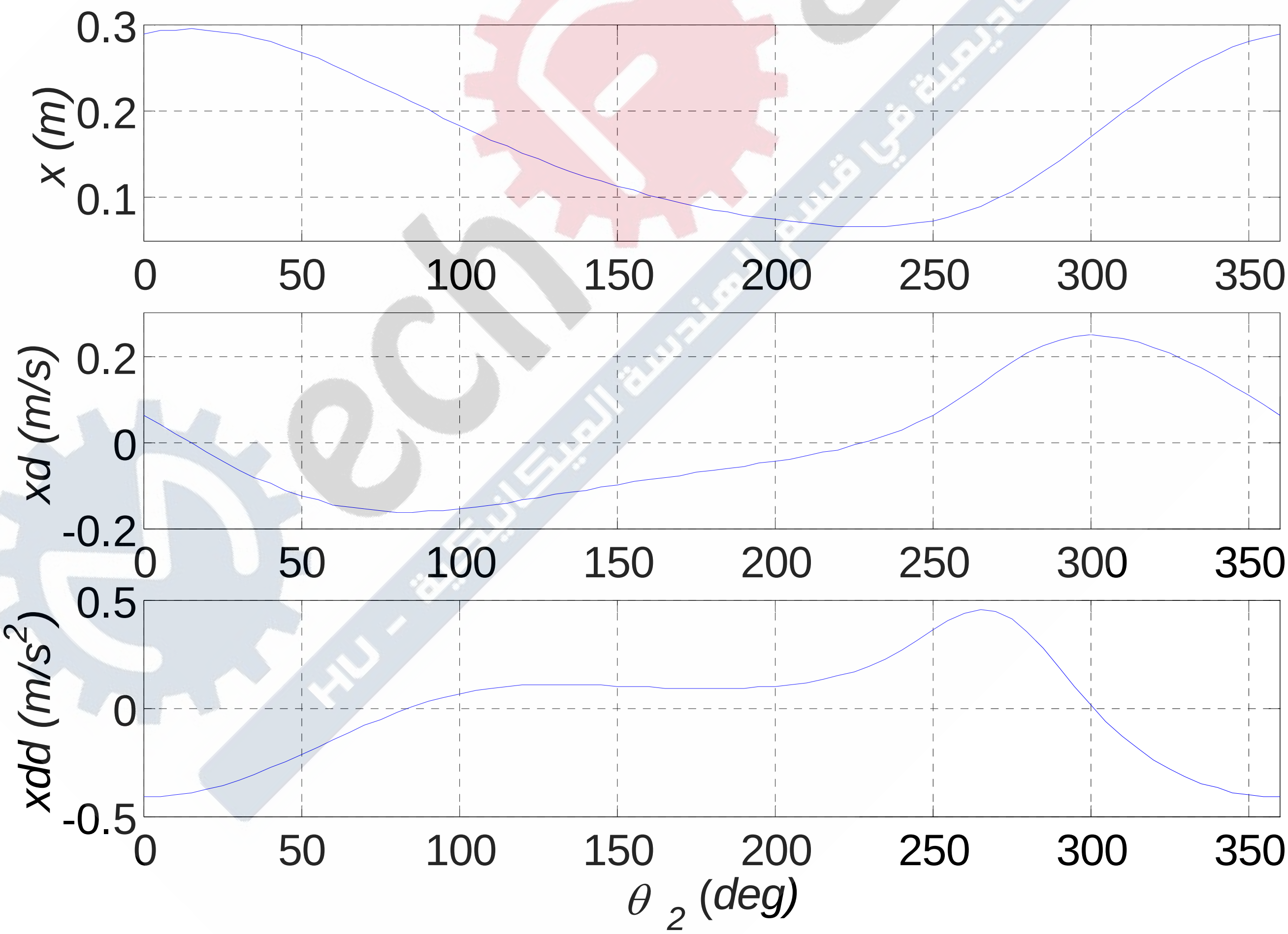


Derivative/Integral Relationships

When one variable is the derivative of another, recall the relationships from calculus. For example:

$$\dot{x} = \frac{dx}{dt} \qquad\qquad x = x_0 + \int \dot{x}dt$$

$$\ddot{x} = \frac{d\dot{x}}{dt} \qquad\qquad \dot{x} = \dot{x}_0 + \int \ddot{x}dt$$



Input Motion Specification

Up to this point, for full range of motion we have assumed that the input link rotates fully with a given **constant input angular velocity**. Our input constraints have thus been $0^\circ \leq \theta_2 \leq 360^\circ$, ω_2 constant, and $\alpha_2 = 0$. This input motion specification is plotted like this:

Note that we have been plotting calculated results vs. θ_2. Since ω_2 is constant, we have $\theta_2 = \omega_2 t$, so we could just as well plot all results vs. time t, since both θ_2 and t increase steadily (linearly).

This constant ω_2 input specification is fine for mechanisms whose input rotates fully and considering steady-state motion only. Many useful mechanisms have input links that do not rotate fully but travel between joint limits, starting and stopping at zero angular velocity. Why is the previous page's plots unacceptable in this case?

Simplest change – **linear angular velocity** starting and stopping at zero:

We cannot plot vs. θ_2 since it is not increasing linearly – plot vs. t.

What is the weakness of this approach? (Discontinuous acceleration function yields infinite jerk at start, middle, and finish.)

We can fix this with a **trapezoidal input acceleration profile**:

This input motion specification should be fine (trapezoidal input torque is often used for industrial robots), but there are many different zones to handle – what acceleration profile is similar but with a single function?

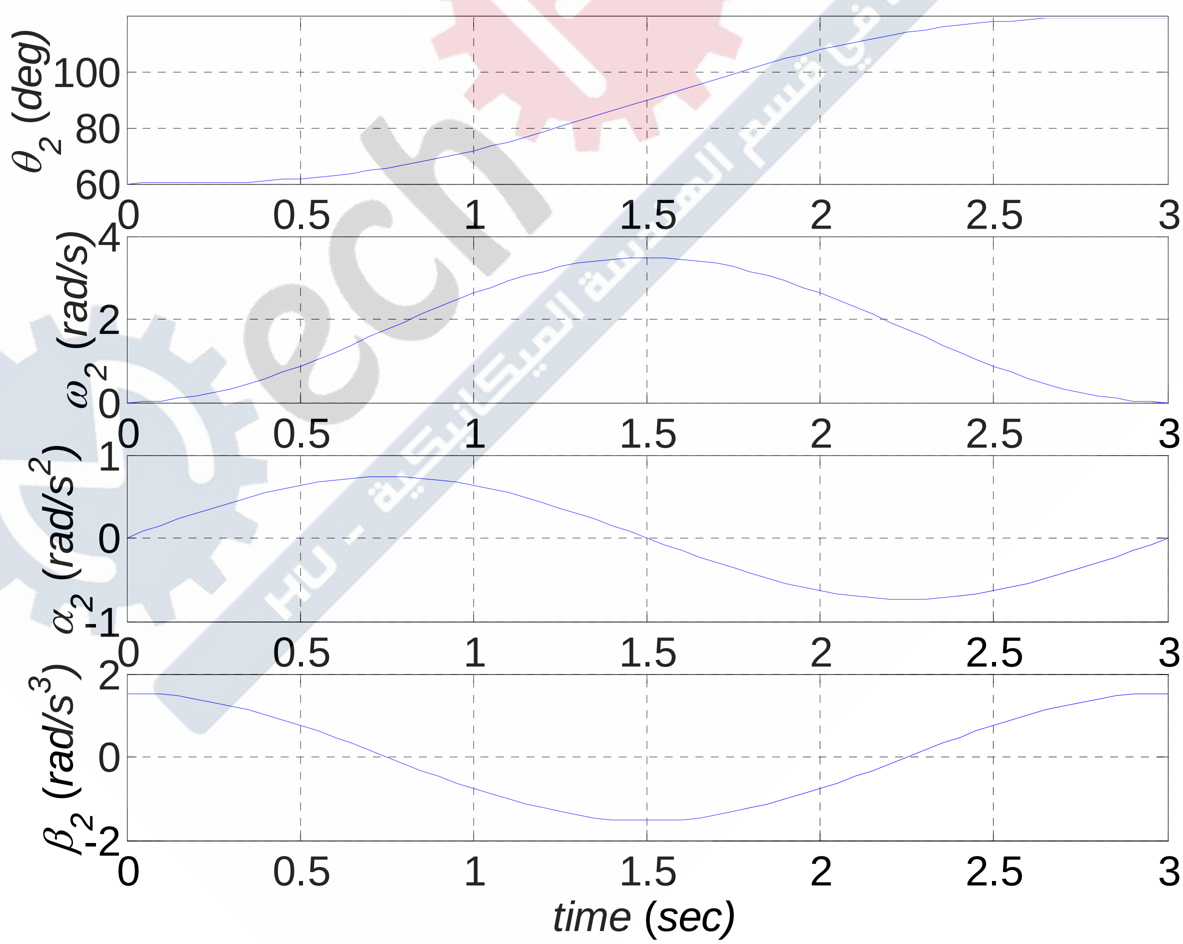
Full-cycloidal function input angle specification

$$\theta_2(t)=\theta_{20}+(\theta_{2F}-\theta_{20})\left(\frac{t}{t_F}-\frac{1}{2\pi}\sin\frac{2\pi t}{t_F}\right)$$

$$\omega_2(t)=\frac{(\theta_{2F}-\theta_{20})}{t_F}\left(1-\cos\frac{2\pi t}{t_F}\right)$$

$$\alpha_2(t)=\frac{2\pi(\theta_{2F}-\theta_{20})}{t_F^2}\left(\sin\frac{2\pi t}{t_F}\right)$$

$$\beta_2(t)=\frac{4\pi^2(\theta_{2F}-\theta_{20})}{t_F^3}\left(\cos\frac{2\pi t}{t_F}\right)$$



Example with $\theta_{20}=60^\circ$, $\theta_{2F}=120^\circ$, and $t_F = 3$ sec.

Dynamics Introduction

Chart:

Kinematics:

translational

rotational

Kinetics:

translational Newton's 2^{nd} Law:

rotational Euler's equation:

Dynamics of a single rigid body in the plane

Rigid body acted on by a system of forces and moments to produce planar motion. What is the first step in analysis? Draw . . .

Free Body Diagram (FBD)

Isolate each rigid body and show the forces and moments acting. This contains all the info needed to write Newton's 2^{nd} Law and Euler's equation.

FBD Simplified FBD

MAD (mass-acceleration diagram)

Internal and External Forces and Moments

All internal and external forces and moments must be included on the FBD.

External forces/moments:

Internal forces/moments:

Write dynamics equations

Newton's 2nd Law:

Euler's equation:

$\underline{A}_G$ is the linear acceleration of center of gravity – same direction as $\underline{R}$. Different points in rigid body have different linear accelerations. $\underline{\alpha}$ angular acceleration of rigid body. The entire rigid body experiences the same $\underline{\alpha}$.

D'Alembert's Principle

Turn dynamics problem into a statics problem by the inclusion of a fictitious "inertial force" $\underline{F}_0 = -m\underline{A}_G$ and a fictitious "inertial moment" $\underline{M}_0 = -I_G\underline{\alpha}$. "Centrifugal force" $-mr\omega^2$ is an example of an inertial force; it's not really a force but an effect of acceleration and inertia. Subtract RHS of equations, then sum to zero as in statics. We won't use this method, just wanted you to know in case you ran into it somewhere.

$$\underline{R} - m\underline{A}_G = 0$$
$$\underline{R} + \underline{F}_O = 0$$

$$\underline{T} + \underline{r} \times \underline{R} - I_G\underline{\alpha} = 0$$
$$\underline{T} + \underline{r} \times \underline{R} + \underline{M}_O = 0$$

Two Types of Dynamics Problems

Forward Dynamics:

Given the mechanism, external forces and moments, and the applied driving force (or torque), find the resulting mechanism motion and internal joint forces.

Inverse Dynamics:

Given the mechanism, external forces and moments, and the desired mechanism motion, find the required driving force (or torque) and internal joint forces.

4-Bar Linkage Forward Dynamics:

Given $\underline{\tau}_2$ and $\underline{F}_{EXT}, \underline{M}_{EXT}$, find the motion $\theta_2, \theta_3, \theta_4$, $\omega_2, \omega_3, \omega_4$, $\alpha_2, \alpha_3, \alpha_4$ and internal forces $\underline{F}_{ij}$.

4-Bar Linkage Inverse Dynamics:

Given the motion $\theta_2, \theta_3, \theta_4$, $\omega_2, \omega_3, \omega_4$, $\alpha_2, \alpha_3, \alpha_4$, and $\underline{F}_{EXT}, \underline{M}_{EXT}$, find $\underline{\tau}_2$ and internal forces $\underline{F}_{ij}$.

<u>Next lecture:</u> Newton's 2nd Law and Euler's equation require:

translational:	mass	center of gravity	
rotational:		center of gravity	mass moment of inertia
	m	P_G	I_G

Mass, Center of Gravity, Mass Moment of Inertia

$$\sum \underline{F} = m\underline{A}_G \qquad\qquad \sum \underline{M}_G = I_G \underline{\alpha}$$

Translational: mass center of gravity
Rotational: center of gravity mass moment of inertia

Mass

In Newton's 2nd Law $\sum \underline{F} = m\underline{a}$, mass m is the proportionality constant. Mass is measure of translational inertia – resistance to change in motion, Newton's 1st Law. Mass is also measure of storage of translational kinetic energy $KE_T = \frac{1}{2}mv^2$.

Examples for *m*, *CG*, I_G:

System of particles **General rigid body**

Rectangular rigid body

Mass calculation:

System of particles:

General rigid body:

Rectangular rigid body:

<u>Center of Gravity (*CG*, *G*)</u>

Also called center of mass, mass center, centroid

<u>*CG* calculation:</u>

System of particles:

General rigid body:

Rectangular rigid body:

Using an XY coordinate frame centered at the geometric center.

$$\bar{X} = \frac{\int_x xdm}{\int dm} \qquad\qquad \bar{Y} = \frac{\int_y ydm}{\int dm}$$

$$= \frac{\rho}{m}\int_x xdV \qquad\qquad = \frac{\rho}{m}\int_y ydV$$

$$= \frac{\rho}{m}\int_{-b/2}^{b/2} xthdx \qquad\qquad = \frac{\rho}{m}\int_{-h/2}^{h/2} ytbdy$$

$$= \frac{\rho th}{m}\int_{-b/2}^{b/2} xdx \qquad\qquad = \frac{\rho tb}{m}\int_{-h/2}^{h/2} ydy$$

$$= \frac{\rho th}{m}\left.\frac{x^2}{2}\right|_{-b/2}^{b/2} \qquad\qquad = \frac{\rho tb}{m}\left.\frac{y^2}{2}\right|_{-h/2}^{h/2}$$

$$= \frac{\rho th}{2m}\left(\frac{b^2}{4} - \frac{b^2}{4}\right) = 0 \qquad\qquad = \frac{\rho tb}{2m}\left(\frac{h^2}{4} - \frac{h^2}{4}\right) = 0$$

For a homogeneous, regular geometric body, the CG is the geometric center.

Mass Moment of Inertia (I_G) is not the same as **<u>Area moment of inertia</u>** (I_G) for beam bending:

$$I_{Ax} = \int_y y^2 dA \qquad I_{Ay} = \int_x x^2 dA$$

Units: $I_A \equiv m^4$

<u>Mass Moment of Inertia (I_G)</u>

In Euler's equation $\sum \underline{M}_G = I_{GZ}\,\underline{\alpha}$, I is the proportionality constant. I is measure of rotational inertia – resistance to change in motion, Newton's 1st Law. Also, it is a measure of how hard it is to accelerate in rotation about certain axes. I is also measure of storage of rotational kinetic energy $KE_R = \frac{1}{2} I_G \omega^2$.

Units: $I_G \equiv kgm^2$.

Mass Moment of Inertia I_G calculation:

System of particles:

where r_i is the scalar perpendicular distance from the axis to the i^{th} particle. With squaring, all terms will be positive, no there can be no canceling like for *CG*. If first moment is balanced, second moment will be doubled about the *CG*.

General rigid body:

What is the only term that matters for *XY* planar motion?

In the example shown above:

$$I_{ZZG} > I_{YYG} > I_{XXG} \qquad \text{also} \qquad I_{ZZ} > I_{ZZG}$$

Rectangular rigid body:

Using an *XY* coordinate frame centered at the *CG*.

$$I_{ZZG} = \int_{body} \left(x^2 + y^2\right) dm = \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} \left(x^2 + y^2\right) \rho t dx dy$$

$$I_{ZZG} = \rho t \int_{-b/2}^{b/2} \left(x^2 y + \frac{y^3}{3} \Bigg|_{-h/2}^{h/2} \right) dx$$

$$= \rho t \int_{-b/2}^{b/2} \left(x^2 \left(\frac{h}{2} - \frac{-h}{2} \right) + \frac{1}{3} \left(\frac{h^3}{8} - \frac{-h^3}{8} \right) \right) dx$$

$$I_{ZZG} = \rho t \int_{-b/2}^{b/2} \left(h x^2 + \frac{h^3}{12} \right) dx = \rho t \left(\frac{h x^3}{3} + \frac{h^3 x}{12} \Bigg|_{-b/2}^{b/2} \right)$$

$$I_{ZZG} = \rho t \left(\frac{h}{3} \left(\frac{b^3}{8} - \frac{-b^3}{8} \right) + \frac{h^3}{12} \left(\frac{b}{2} - \frac{-b}{2} \right) \right)$$

$$= \rho t \left(\frac{b^3 h}{12} + \frac{b h^3}{12} \right) = \frac{\rho t b h}{12} \left(b^2 + h^2 \right)$$

$$I_{ZZG} = \frac{m}{12} \left(b^2 + h^2 \right) \qquad \text{(because } m = \rho V = \rho t b h\text{)}$$

Units: mass times distance squared, kgm^2.

Checks with result given in the textbook.

How do we find mass moments of inertia in the real-world?

- look up in tables
- CAD package such as SolidEdge

Parallel Axis Theorem

The mass moment of inertia through the CG is related to mass moments of inertia of parallel axes through different points as follows:

where d is the scalar distance separating the axis of interest from the axis through the CG. Notice I_{ZZG} is a small as it can get; any I_{ZZ} must be greater, due to the term md^2, which is always positive.

Parallel axis theorem example:

Rectangular rigid body:

$$I_{ZZ} = \frac{m}{12}\left(b^2 + h^2\right) + m\left(\frac{b^2}{4} + \frac{h^2}{4}\right)$$

$$= m\left(\frac{b^2}{12} + \frac{b^2}{4} + \frac{h^2}{12} + \frac{h^2}{4}\right)$$

$$= m\left(\frac{b^2}{3} + \frac{h^2}{3}\right)$$

$$= \frac{m}{3}\left(b^2 + h^2\right)$$

Agrees with result given in dynamics textbooks.

Single Rotating Link Inverse Dynamics

Generic Mechanism Inverse Dynamics Analysis Statement:
Given the mechanism, external forces and moments, and the desired mechanism motion, find the required driving force (or torque) and internal joint forces.

Single Rotating Link Inverse Dynamics Analysis

Step 1. **Position, Velocity, and Acceleration Analyses** must first be complete.

Step 2. **Draw the Diagrams:**

Physical Dynamics Diagram:

Free Body Diagram (FBD):

Step 3. **State the problem:**

Step 4. **Derive the Newton-Euler Dynamics Equations**.

Newton's 2^{nd} Law:

Euler's Equation:

Count # of unknowns and # of equations:

Step 5. **Derive *XYZ* scalar equations** from the vector equations.

Written in matrix form:

Step 6. **Solve for the unknowns**

Actually, we don't need matrix solution; the first two equations are decoupled and the solution is straight-forward:

Step 7. **Calculate Shaking Force and Moment**

After the inverse dynamics problem is solved, we can calculate the vector shaking force and moment, which is the force/moment reaction on the ground link due to the mechanism, motion, and external loads:

Terms for the inverse dynamics equations

The inverse dynamics problem has been solved analytically for the single rotating link. Now, how do we calculate the various terms that appear in the dynamics equations? These all must be derived from given information.

$$\underline{A}_G = \begin{Bmatrix} A_{GX} \\ A_{GY} \end{Bmatrix} =$$

$$\underline{F}_E = \begin{Bmatrix} F_{EX} \\ F_{EY} \end{Bmatrix} =$$

$$\underline{r}_{12} = \begin{Bmatrix} r_{12X} \\ r_{12Y} \end{Bmatrix} =$$

$$\underline{r}_E = \begin{Bmatrix} r_{EX} \\ r_{EY} \end{Bmatrix} =$$

$$I_{GZ} =$$

Single rotating link inverse dynamics example:

Given: $L = 1\ m$, $h = 0.1\ m$, $m = 2\ kg$, $\omega = 100\ rad/s$, $\alpha = 0$, F_E=150 N, $\phi_E = 0$ (constant relative to horizontal), M_E=0 Nm.

Calculated terms: $\|r_{12}\| = \|r_E\| = 0.5m \quad I_{GZ} = 0.17 kgm^2$

$$\begin{aligned} A_{Gx} &= 4330 \\ A_{Gy} &= -2500 \end{aligned} \quad \frac{m}{s^2}$$

Snapshot Analysis (one input angle)

At $\theta = 150^\circ$, given this link, motion, and external force, calculate F_{12X}, F_{12Y}, τ and $\underline{F}_S, \underline{M}_S$ for this instant (snapshot).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.250 & 0.433 & 1 \end{bmatrix} \begin{Bmatrix} F_{12X} \\ F_{12Y} \\ \tau \end{Bmatrix} = \begin{Bmatrix} 8510 \\ -4980 \\ 37.5 \end{Bmatrix}$$

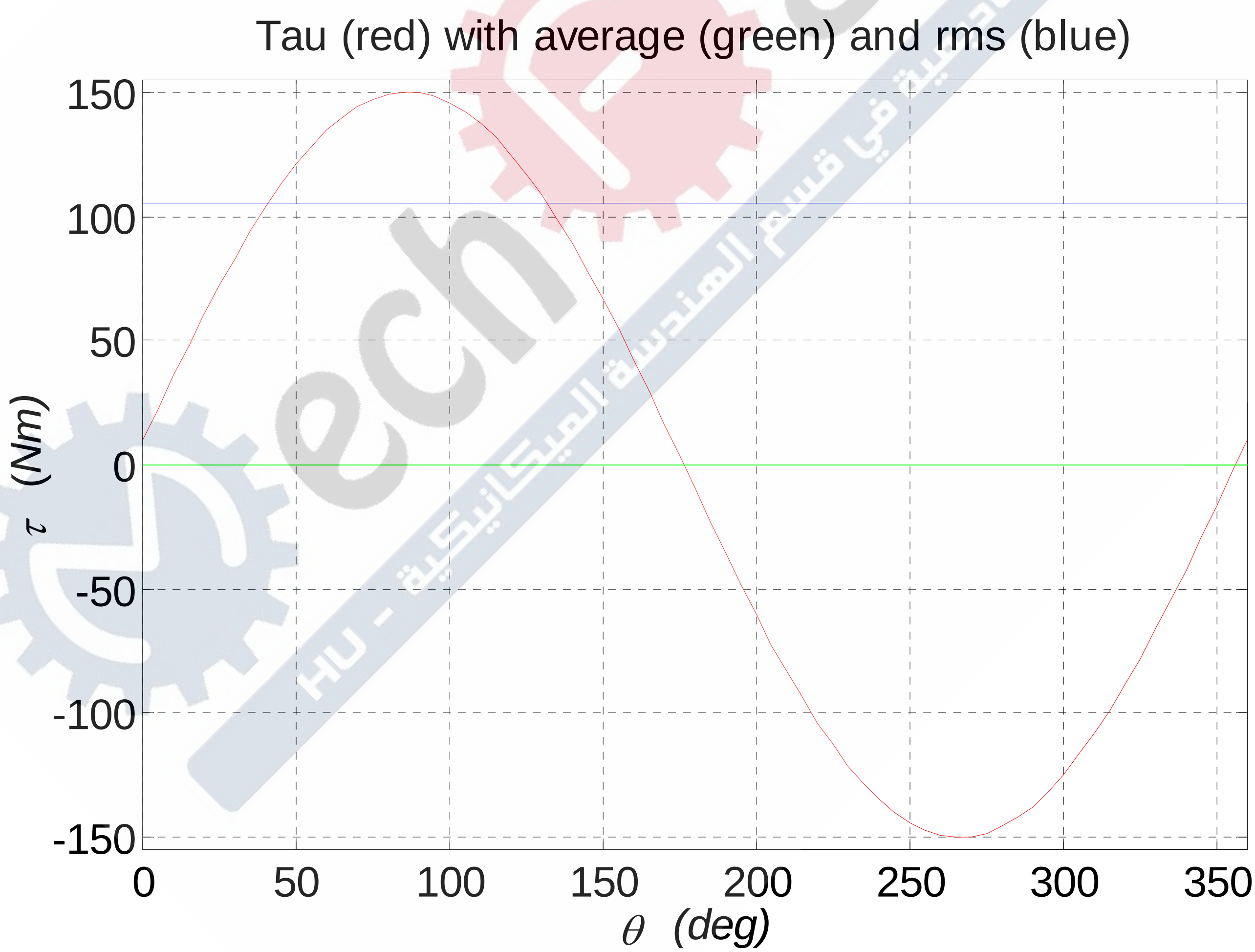
$$\begin{Bmatrix} F_{12X} \\ F_{12Y} \\ \tau \end{Bmatrix} = \begin{Bmatrix} 8510 \\ -4980 \\ 66.5 \end{Bmatrix} N, Nm$$

$$\underline{F}_S = \underline{F}_{21} = -\underline{F}_{12} = \begin{Bmatrix} -8510 \\ 4980 \end{Bmatrix} N$$

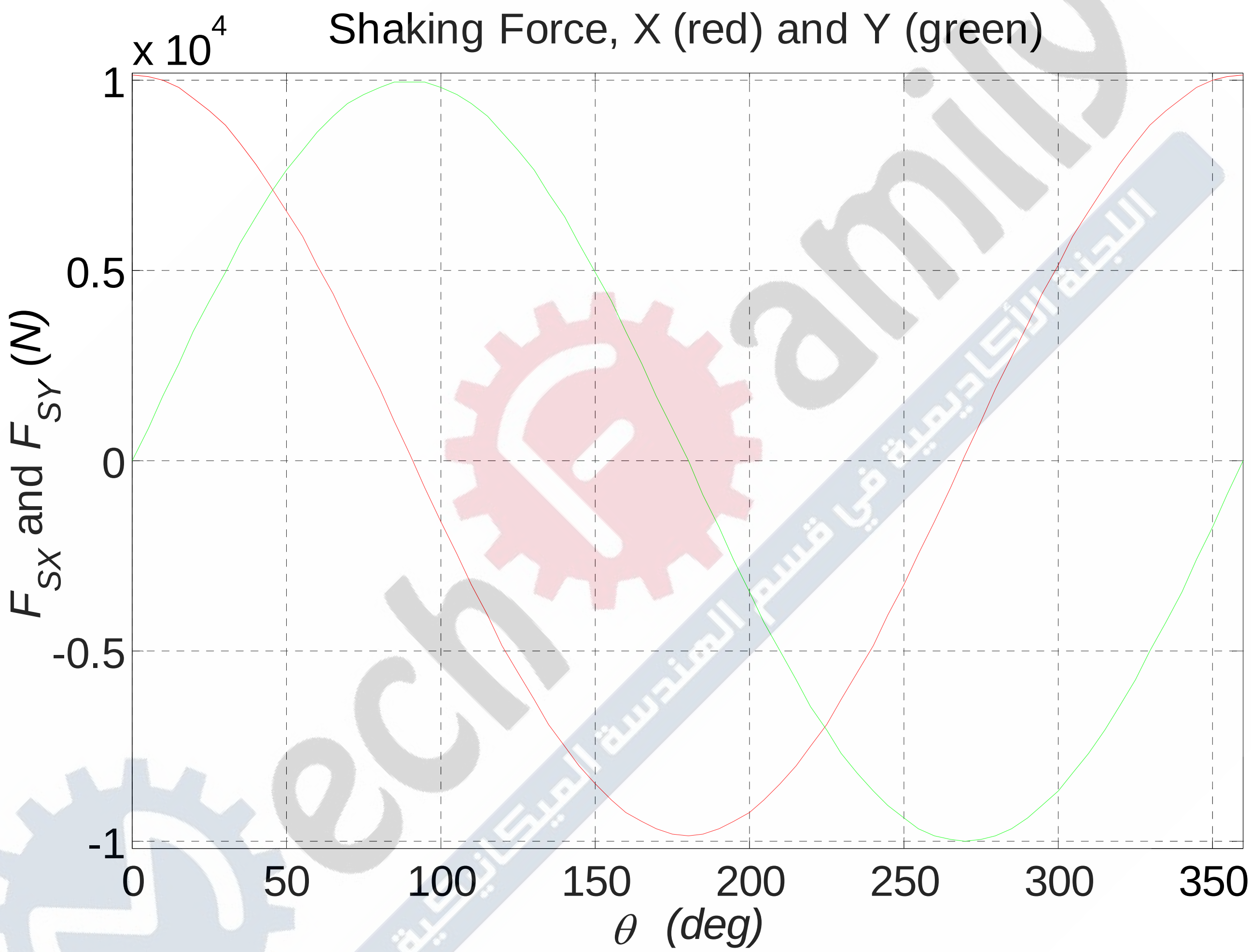
$$\underline{M}_S = -\underline{\tau} = -66.5\hat{k}\ Nm$$

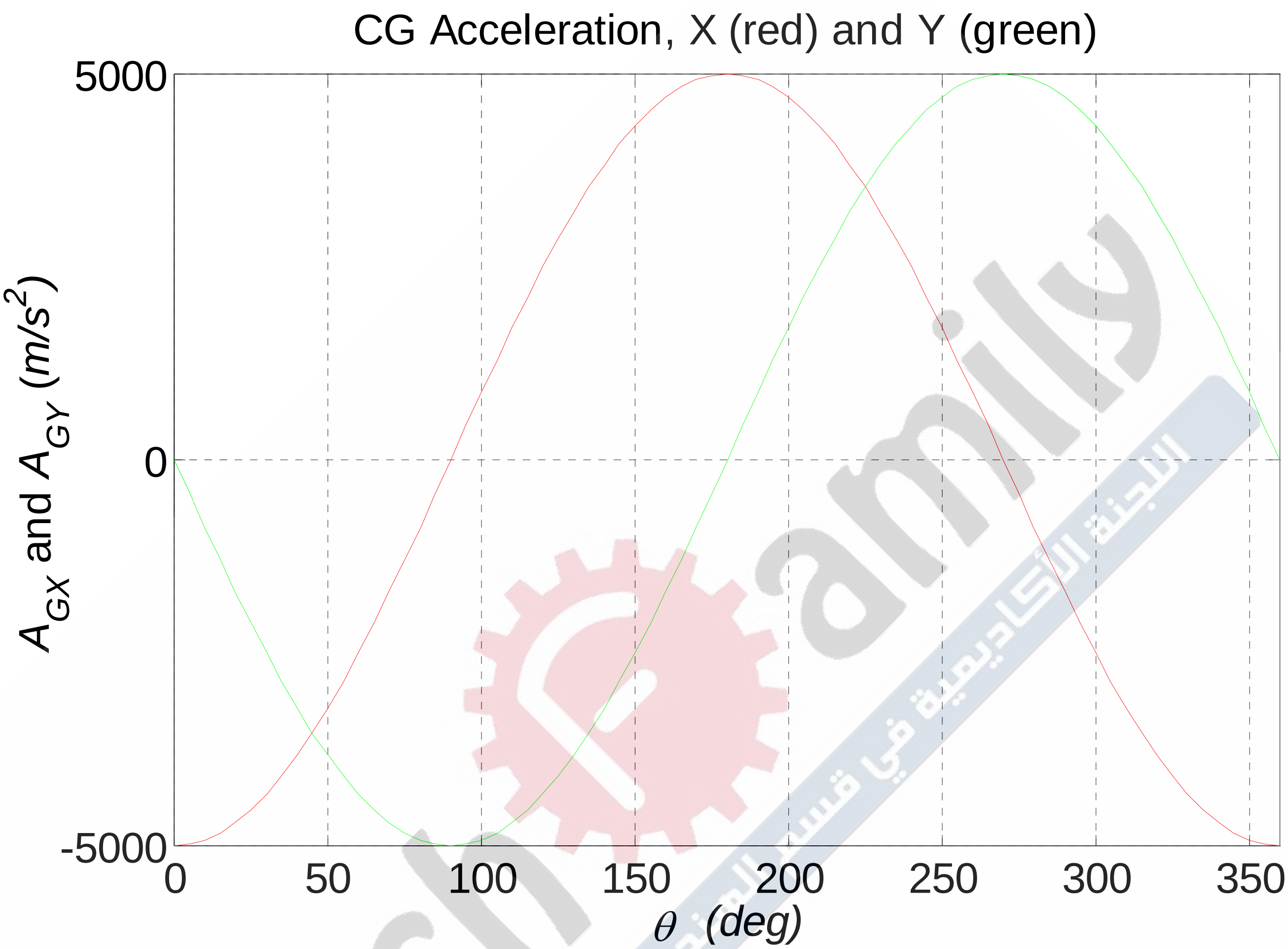
Full-Range-Of-Motion (F.R.O.M.) Analysis

A more meaningful result from inverse dynamics analysis is to report the unknowns for the entire range of mechanism motion. The plot below gives the required driving torque $\underline{\tau}$ (*Nm*, red) for all $0^\circ \le \theta \le 360^\circ$, assuming the given ω is constant, for the same example from the previous page. This shows the torque that must be supplied by an external DC servomotor to cause the specified motion. Also plotted is the average torque (green) $\tau_{AVG} = 0$ and the root-mean-square (RMS) torque value (blue) $\tau_{RMS} = 106.1$ *Nm*.



The plots below give the Shaking Force $\underline{F}_S$ and CG translational acceleration results, respectively, for all $0^\circ \le \theta \le 360^\circ$. In both plots, the X components are red and the Y green.





The Shaking Moment $\underline{M}_S$ is merely the negative of the driving torque $\underline{\tau}$ plot shown previously and hence is not shown separately. Is the static loading (mg) significant?

Four-Bar Mechanism Inverse Dynamics

Generic Mechanism Inverse Dynamics Analysis Statement:

Given the mechanism, external forces and moments, and the desired mechanism motion, find the required driving force (or torque) and internal joint forces.

Four-Bar Mechanism Inverse Dynamics Analysis

First, can we simplify and solve the problem link-by-link, like the single rotating link? Count # of unknowns and # of equations:

Step 1. **Position, Velocity, and Acceleration Analyses** must first be complete.

Step 2. **Draw the Diagrams:**
Physical Dynamics Diagram:

Free Body Diagrams (FBDs):

$\underline{F}_{ij}$:

$\underline{r}_{ij}$:

<u>Step 3.</u> **<u>State the problem:</u>**

<u>Step 4.</u> **<u>Derive the Newton-Euler Dynamics Equations</u>**.

Newton's 2nd Law:

Euler's Equation:

Count # of unknowns and # of equations:

Step 5. **Derive *XYZ* scalar equations** from the vector equations.

Write these equations in matrix/vector form:

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ r_{12Y} & -r_{12X} & -r_{32Y} & r_{32X} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & r_{23Y} & -r_{23X} & -r_{43Y} & r_{43X} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & r_{34Y} & -r_{34X} & -r_{14Y} & r_{14X} & 0 \end{bmatrix} \begin{Bmatrix} F_{21X} \\ F_{21Y} \\ F_{32X} \\ F_{32Y} \\ F_{43X} \\ F_{43Y} \\ F_{14X} \\ F_{14Y} \\ \tau_2 \end{Bmatrix} = \begin{Bmatrix} m_2 A_{G2X} \\ m_2\left(A_{G2Y}+g\right) \\ I_{G2Z}\alpha_2 \\ m_3 A_{G3X} - F_{E3x} \\ m_3\left(A_{G3Y}+g\right) - F_{E3y} \\ I_{G3Z}\alpha_3 - r_{E3x}F_{E3y} + r_{E3y}F_{E3x} - M_{E3} \\ m_4 A_{G4X} - F_{E4x} \\ m_4\left(A_{G4Y}+g\right) - F_{E4y} \\ I_{G4Z}\alpha_4 - r_{E4x}F_{E4y} + r_{E4y}F_{E4x} - M_{E4} \end{Bmatrix}$$

$$[A]\{v\} = \{b\}$$

Coefficient matrix $[A]$ dependent on geometry (kinematics solutions). RHS $\{b\}$ dependent on inertial terms, gravity, and given external forces and moments.

Step 6. **Solve for the unknowns**

Simultaneous matrix solution: $\{v\} = [A]^{-1}\{b\}$

Actually, using Gaussian elimination is more efficient and robust.

Solution to internal forces and input torque are contained in the components of $\{v\}$.

Step 7. **Calculate Shaking Force and Moment**

After the basic inverse dynamics problem is solved, we can calculate the vector shaking force and moment, which is the force/moment reaction on the ground link due to the motion:

Details for the general four-bar mechanism model

The inverse dynamics problem has been derived analytically for the four-bar mechanism. Now, how do we calculate the various terms that appear in the dynamics equations? These all must be derived from given information. See Fig. P11-2. Let us do link 3 terms (next page). Here is the general link 3 diagram for these derivations:

<u>Link 3 details:</u>

$$\underline{r}_{23} = \begin{Bmatrix} r_{23X} \\ r_{23Y} \end{Bmatrix} =$$

$$\underline{r}_{43} = \begin{Bmatrix} r_{43X} \\ r_{43Y} \end{Bmatrix} =$$

$$\underline{A}_{G3} = \begin{Bmatrix} A_{G3X} \\ A_{G3Y} \end{Bmatrix} =$$

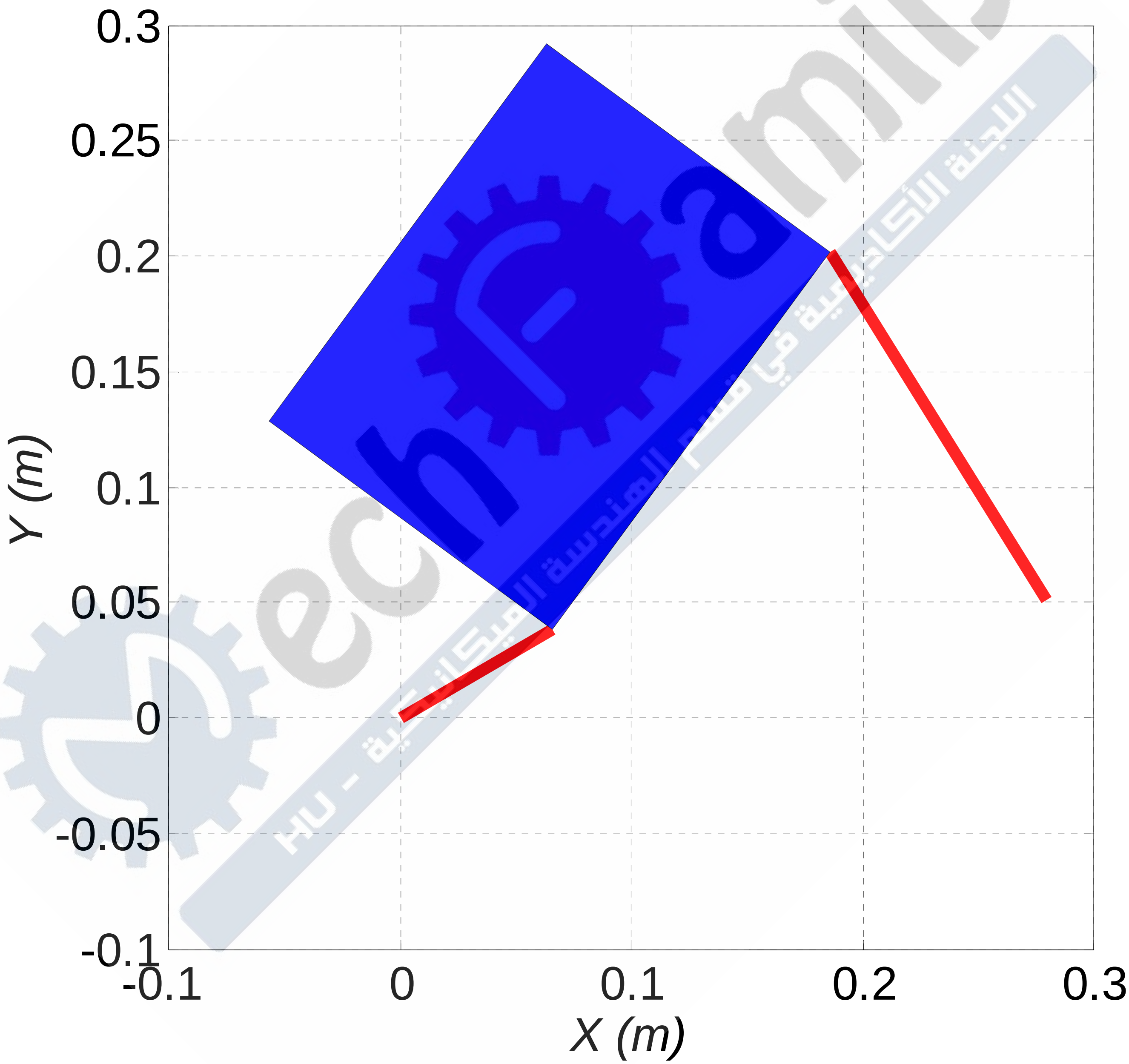
$$\underline{F}_{P_3} = \begin{Bmatrix} F_{P_{3X}} \\ F_{P_{3Y}} \end{Bmatrix} =$$

$$\underline{R}_{P_3} = \begin{Bmatrix} R_{P_{3X}} \\ R_{P_{3Y}} \end{Bmatrix} =$$

$\underline{M}_{E3}$ = given

<u>Figure for Term Example 1 Inverse Dynamics Example starting on the next page:</u>

The coupler link 3 is a rectangle of dimensions 8” x 6” x 0.5”. The triangle tip we have been using all along in Term Example 1 is actually the *CG*; of the actual rectangular link for inverse dynamics.



Four-bar mechanism inverse dynamics example:

This is the mechanism from Term Example 1 (open branch), with one crucial difference: the input angular velocity was too low for interesting dynamics, so I changed it from $\omega_2 = \pi$ to $\omega_2 = 20$ *rad/s*.

Given $r_1 = 0.284$, $r_2 = 0.076$, $r_3 = 0.203$, $r_4 = 0.178$, $R_{G3} = 0.127$ *m*, and $\theta_1 = 10.3^\circ$, $\theta_2 = 30^\circ$, $\theta_3 = 53.8^\circ$, $\theta_4 = 121.7^\circ$, $\delta_3 = 36.9^\circ$; $\omega_2 = 20$, $\omega_3 = -8.09$, $\omega_4 = -3.73$ *rad/s*; $\alpha_2 = 0$, $\alpha_3 = 8.65$, $\alpha_4 = 244.4$ *rad/s*^{2}. This is the open branch of the position, velocity, and acceleration example (Term Example 1).

All moving links are wood, with mass density $\rho = 0.03$ lb_{m}/in^3. Links 2 and 4 have rectangular dimensions 0.75 by 0.50 by r_i (*in*); link 3 has rectangular dimensions 8 by 6 by 0.5 (*in*), as shown on the previous page. The calculated inertia parameters are $m_2 = 0.015$, $m_3 = 0.327$, $m_4 = 0.036$ *kg* and $I_{G2Z} = 7.9 \times 10^{-6}$, $I_{G3Z} = 1.8 \times 10^{-3}$, $I_{G4Z} = 9.5 \times 10^{-5}$ kgm^2. All external forces and moments are zero but gravity is included.

Snapshot Analysis (one input angle)

At $\theta_2 = 30^\circ$, given this mechanism and motion, calculate the four vector internal joint forces, the driving torque $\underline{\tau}_2$, and the shaking force and moment $\underline{F}_S, \underline{M}_S$ for this instant (snapshot).

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.019 & 0.033 & -0.019 & 0.033 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -0.127 & -0.002 & -0.037 & 0.122 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0.076 & 0.047 & 0.076 & 0.047 & 0 \end{bmatrix} \begin{Bmatrix} F_{21x} \\ F_{21y} \\ F_{32x} \\ F_{32y} \\ F_{43x} \\ F_{43y} \\ F_{14x} \\ F_{14y} \\ \tau_2 \end{Bmatrix} = \begin{Bmatrix} -0.202 \\ 0.034 \\ 0 \\ -8.955 \\ -4.497 \\ 0.015 \\ -0.638 \\ -0.095 \\ 0.0233 \end{Bmatrix}$$

Solution by Gaussian elimination or: $\{v\} = [A]^{-1}\{b\}$

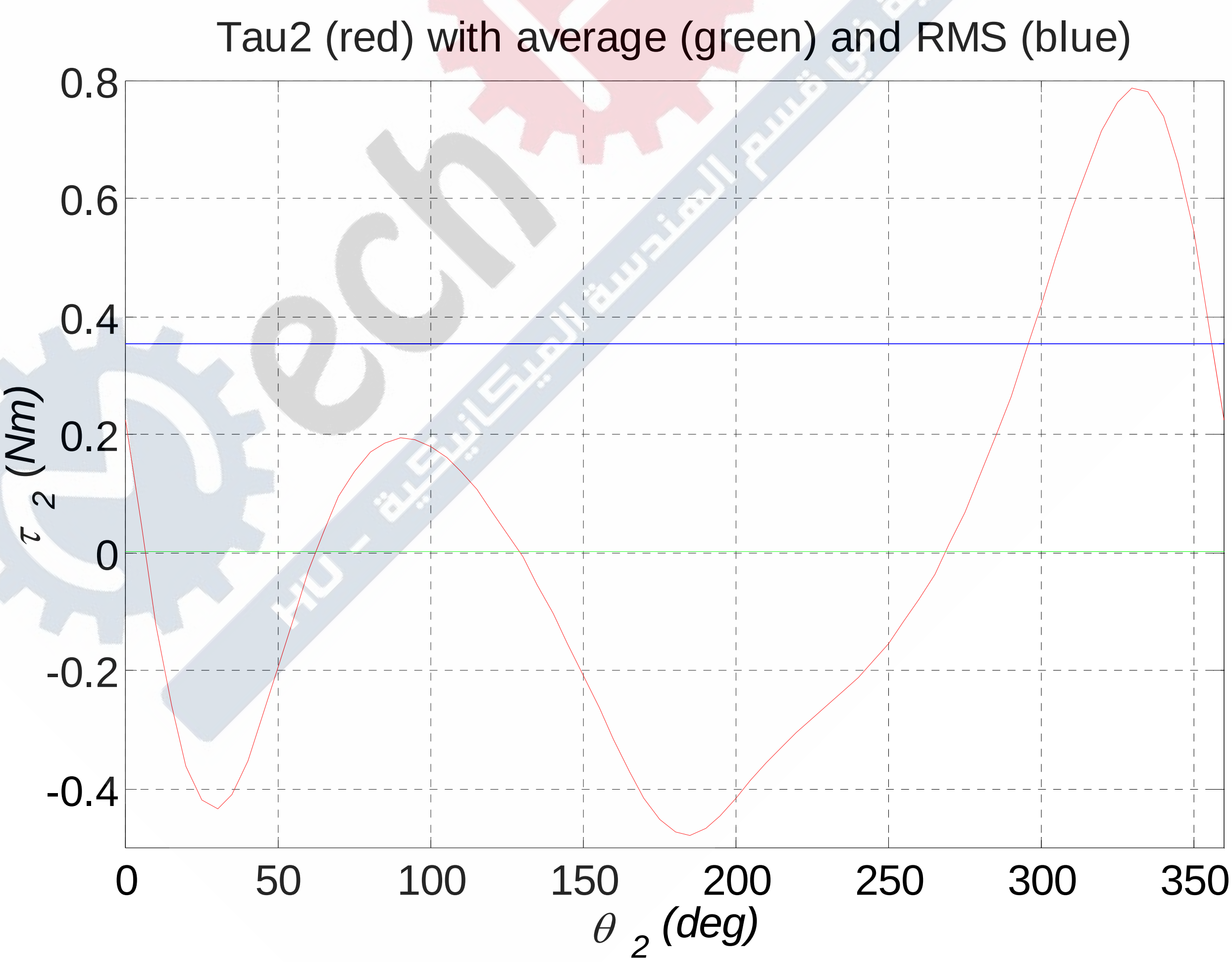
Snapshot Answer:

$$\{v\} = \begin{Bmatrix} F_{21x} \\ F_{21y} \\ F_{32x} \\ F_{32y} \\ F_{43x} \\ F_{43y} \\ F_{14x} \\ F_{14y} \\ \tau_2 \end{Bmatrix} = \begin{Bmatrix} 6.20 \\ 10.08 \\ 5.99 \\ 10.11 \\ -2.96 \\ 5.61 \\ -3.60 \\ 5.52 \\ -0.43 \end{Bmatrix} N, Nm$$

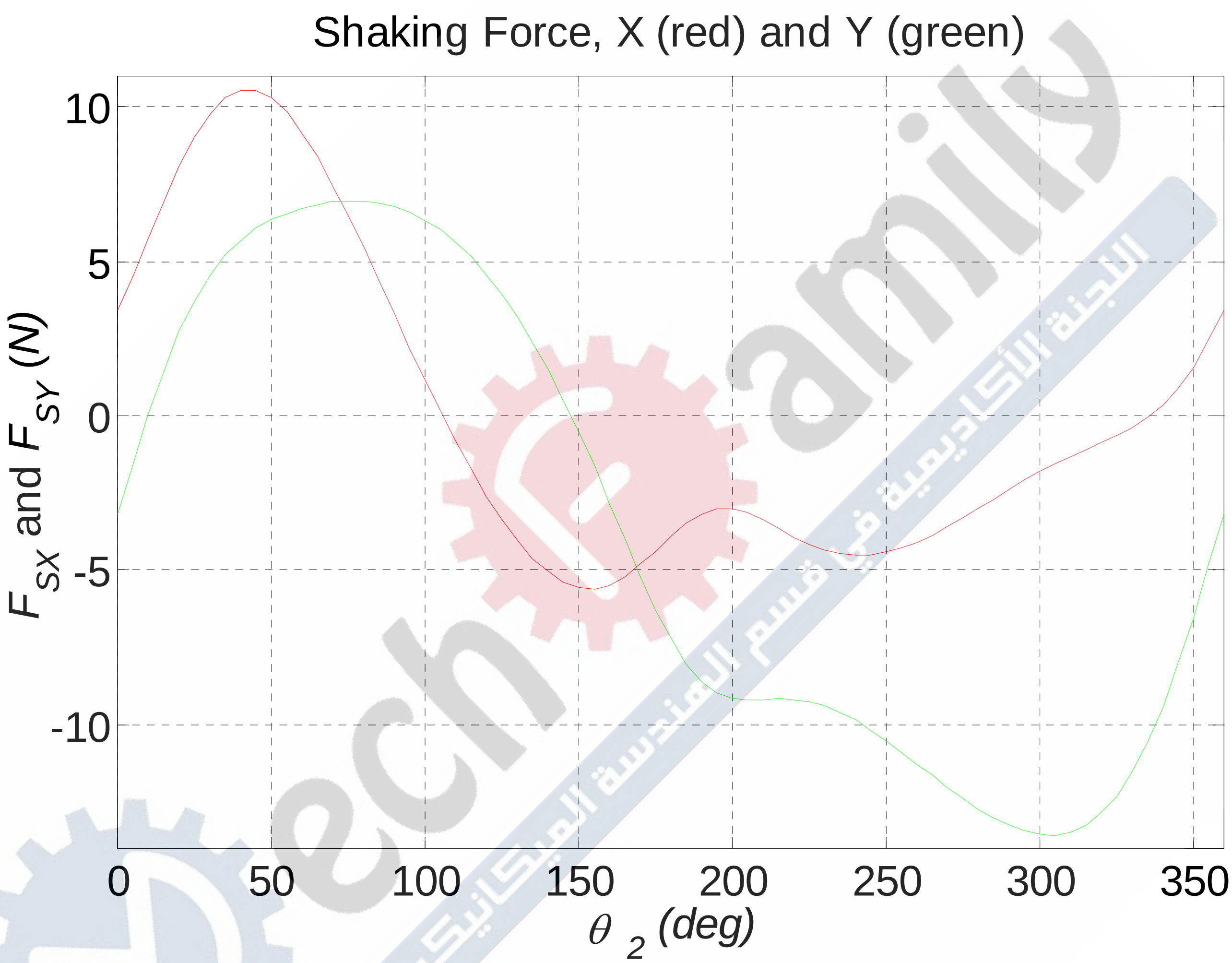
$$\underline{F}_S = \begin{Bmatrix} 9.80 \\ 4.56 \end{Bmatrix} N \qquad \underline{M}_S = -1.68\hat{k} \ Nm$$

Full-Range-Of-Motion (F.R.O.M.) Analysis

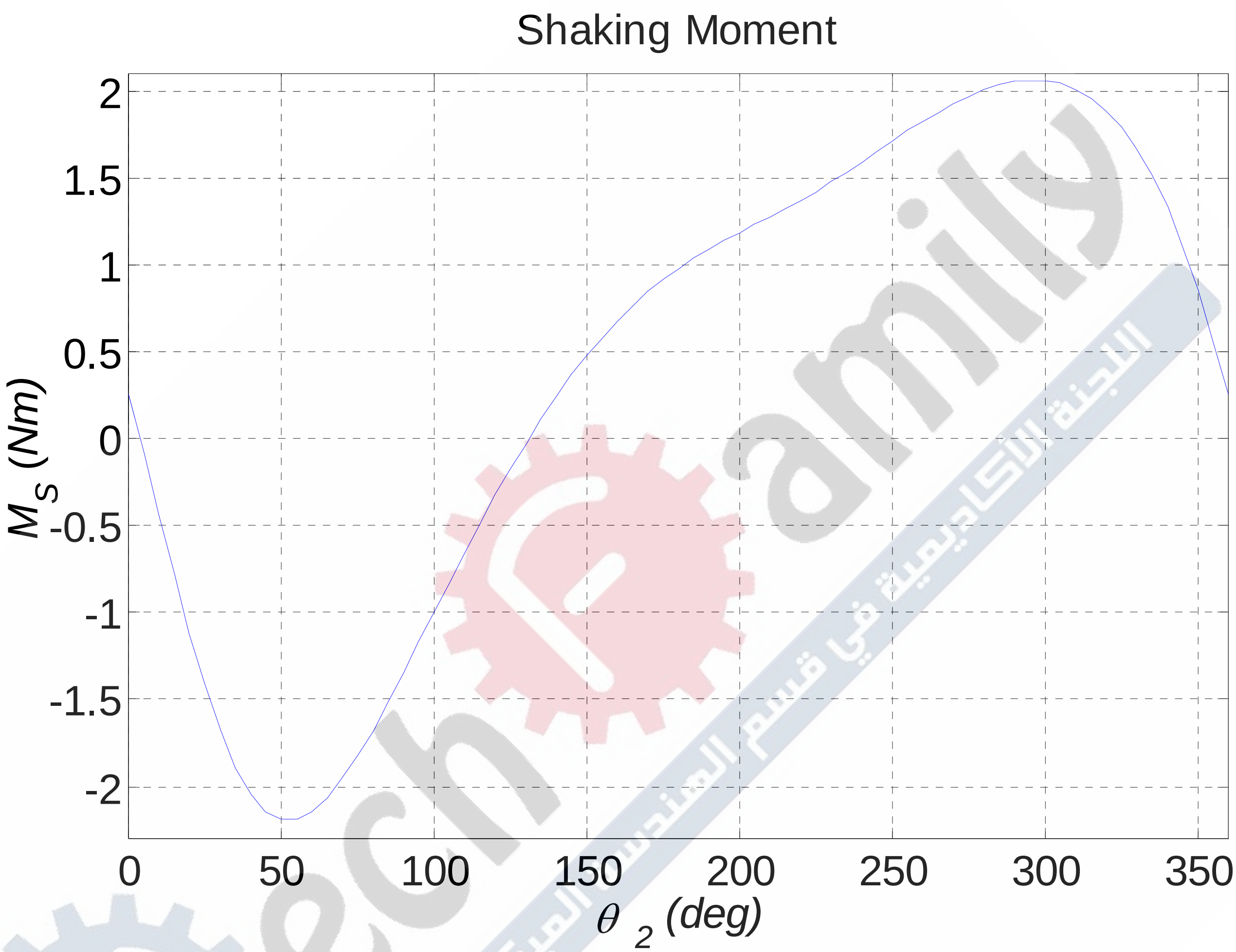
A more meaningful result from inverse dynamics analysis is to report the unknowns for the entire range of mechanism motion. The plot below gives the required driving torque $\underline{\tau}_2$ (Nm) for all $0^\circ \le \theta_2 \le 360^\circ$, for the Term Example 1 mechanism, assuming the given $\omega_2 = 20\ rad/s$ is constant (**Remember:** this has been changed from the $\omega_2 = \pi\ rad/s$ in the kinematics examples!). This plot shows the torque (red) that must be supplied in all configurations by an external DC servomotor to cause the specified motion. Also plotted is the average torque (green) $\tau_{2AVG} = 0$ and the root-mean-square torque value (blue) $\tau_{2RMS} = 0.36\ Nm$.



The plots below give the shaking force $\underline{F}_S$ (N) and shaking moment $\underline{M}_S$ (Nm) results, respectively, for all $0^\circ \le \theta_2 \le 360^\circ$. In the force plot, the X component is red and the Y green.



In the shaking moment plot, there is only the *Z* component:



Slider-Crank Mechanism Inverse Dynamics

This problem is very similar to the four-bar mechanism inverse dynamics problem. In fact, links 2 and 3 are handled identically!

Step 1. **Position, Velocity, and Acceleration Analyses** must first be complete.

Step 2. **Draw the Diagrams:**
Physical Dynamics Diagram:

Free Body Diagrams (FBDs):

$\underline{F}_{ij}$: internal force of link i acting on link j

$\underline{r}_{ij}$: moment arm pointing to link i from the CG of link j

Step 3. **State the problem:**

Step 4. **Derive the Newton-Euler Dynamics Equations**.

Again, links 2 and 3 are identical so let us focus on link 4, the slider. There are two kinematic constraints on the slider:

Newton's 2^{nd} Law:

Euler's Equation:

Count # of unknowns and # of equations: We need an additional equation:

Step 5. **Derive *XYZ* scalar equations** from the vector equations and beam these equations into matrix/vector form. Substitute the friction constraint to eliminate one unknown (F_{14X}); also eliminate one equation ($\sum \underline{M}_{G4} = I_{G4Z}\alpha_4$).

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ r_{12Y} & -r_{12X} & -r_{32Y} & r_{32X} & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & r_{23Y} & -r_{23X} & -r_{43Y} & r_{43X} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & \pm\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_{21X} \\ F_{21Y} \\ F_{32X} \\ F_{32Y} \\ F_{43X} \\ F_{43Y} \\ F_{14Y} \\ \tau_2 \end{Bmatrix} = \begin{Bmatrix} m_2 A_{G2X} \\ m_2\left(A_{G2Y} + g\right) \\ I_{G2Z}\alpha_2 \\ m_3 A_{G3X} - F_{E3x} \\ m_3\left(A_{G3Y} + g\right) - F_{E3y} \\ I_{G3Z}\alpha_3 - r_{E3x}F_{E3y} + r_{E3y}F_{E3x} - M_{E3} \\ m_4 A_{G4X} - F_{E4x} \\ m_4 g - F_{E4y} \end{Bmatrix}$$

$$[A]\{v\} = \{b\}$$

Coefficient matrix $[A]$ dependent on geometry (kinematics solutions). Always choose proper sign of μ to be opposite to the current $\dot{x}$ direction. RHS $\{b\}$ dependent on inertial and statics terms.

Step 6. **Solve for the unknowns**

Simultaneous matrix solution:

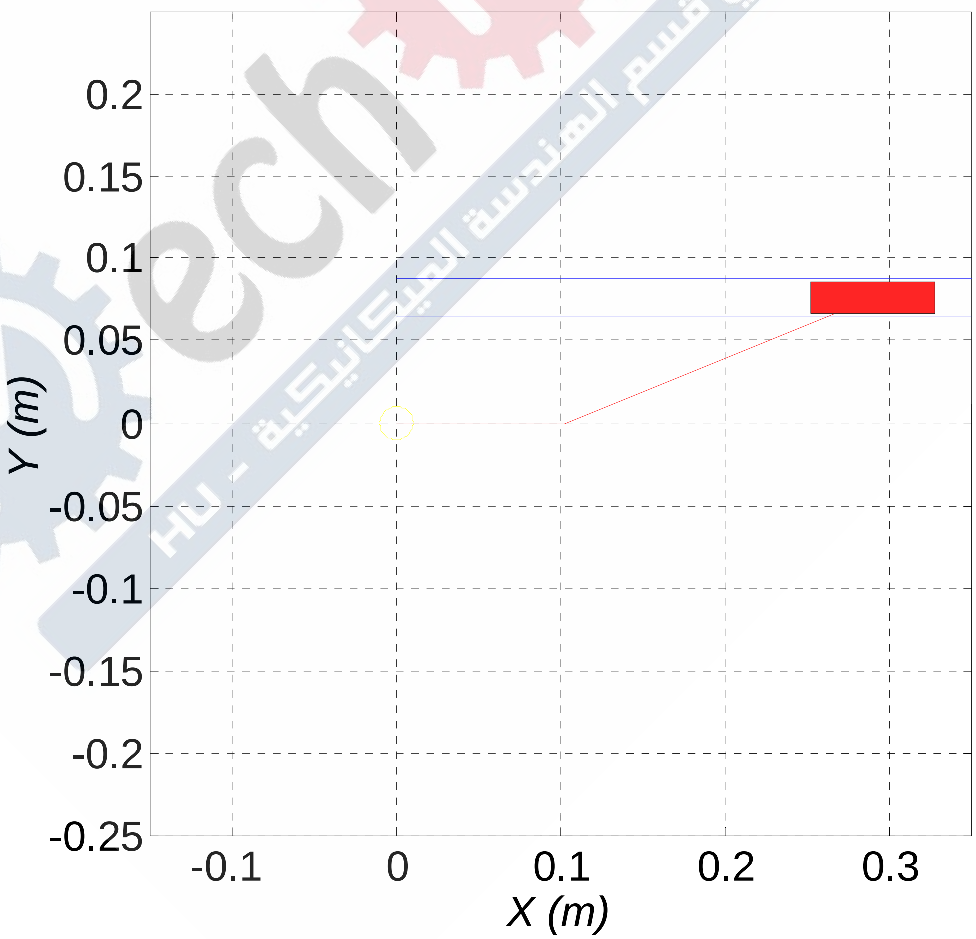
$$\{v\} = [A]^{-1}\{b\}$$

Actually, using Gaussian elimination is more efficient and robust. Solution to internal forces and input torque contained in the components of $\{v\}$.

Step 7. **Calculate Shaking Force and Moment**

After the basic inverse dynamics problem is solved, we can calculate the vector shaking force and moment, which is the force/moment reaction on the ground link due to the motion.

Figure for example starting on the next page: The slider-crank mechanism is shown at the starting (and ending) position, with zero input angle θ_2.



Slider-crank mechanism inverse dynamics example:

This is the mechanism from Term Example 2 (right branch only), in this case keeping the low input angular velocity $\omega_2 = \pi/2$ *rad/s* so the previous snapshots and full-range-of-motion results still apply.

Given r_2 = 0.102, r_3 = 0.203, h = 0.076 m, and $\theta_2 = 30^\circ$, $\theta_3 = 7.2^\circ$, x = 0.290 m; and $\omega_2 = \pi/2$, $\omega_3 = -0.686$ *rad/s*, $\dot{x} = -0.062$ *m/s*; $\alpha_2 = 0$, $\alpha_3 = 0.681$ *rad/s*^{2}, $\ddot{x} = -0.329$ *m/s*^{2}. This is the right branch of the position, velocity, and acceleration example (Term Example 2).

All moving links are wood, with mass density $\rho = 0.03$ (lb_m/in^3). Links 2 and 3 have rectangular dimensions 0.75 by 0.50 by r_i (*in*); link 4 has rectangular dimensions 0.75 by 0.50 by 3 (*in*). The calculated inertia parameters are m_2 = 0.020, m_3 = 0.041, m_4 = 0.015, (*kg*) and I_{G2Z} = 1.819e-005, I_{G3Z} = 1.418e-004 (*kgm*^{2}). There is a constant external force of 1 N acting at the center of the piston, directed horizontally to the left; gravity is included but all other external forces and moments are zero. We assume μ = 0.2 (coefficient of friction between piston and wall);

Snapshot Analysis (one input angle)

At $\theta_2 = 30^\circ$, given this mechanism and motion, calculate the four vector internal joint forces, the driving torque $\underline{\tau}_2$, and the shaking force and moment $\underline{F}_S, \underline{M}_S$ for this instant (snapshot).

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.025 & 0.044 & -0.025 & 0.044 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -0.013 & 0.101 & -0.013 & 0.101 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_{21X} \\ F_{21Y} \\ F_{32X} \\ F_{32Y} \\ F_{43X} \\ F_{43Y} \\ F_{14Y} \\ \tau_2 \end{Bmatrix} = \begin{Bmatrix} -0.002 \\ 0.199 \\ 0 \\ -0.011 \\ 0.398 \\ 0.0001 \\ 0.995 \\ 0.150 \end{Bmatrix}$$

Solution by Gaussian elimination or: $\{v\} = [A]^{-1}\{b\}$

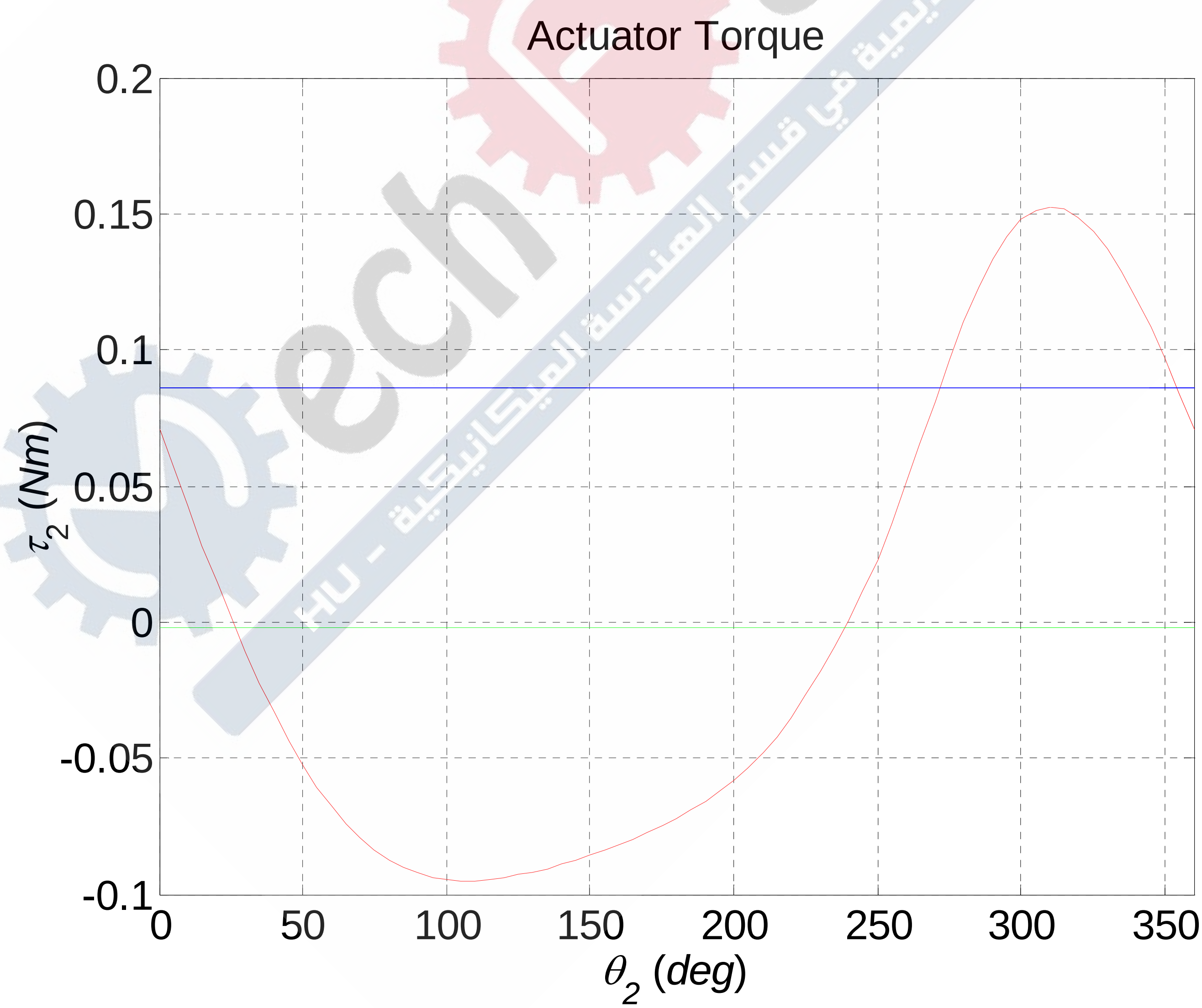
Snapshot Answer:

$$\{v\} = \begin{Bmatrix} F_{21X} \\ F_{21Y} \\ F_{32X} \\ F_{32Y} \\ F_{43X} \\ F_{43Y} \\ F_{14Y} \\ \tau_2 \end{Bmatrix} = \begin{Bmatrix} -0.935 \\ -0.517 \\ -0.938 \\ -0.318 \\ -0.949 \\ 0.081 \\ 0.231 \\ -0.011 \end{Bmatrix} (N,\ Nm)$$

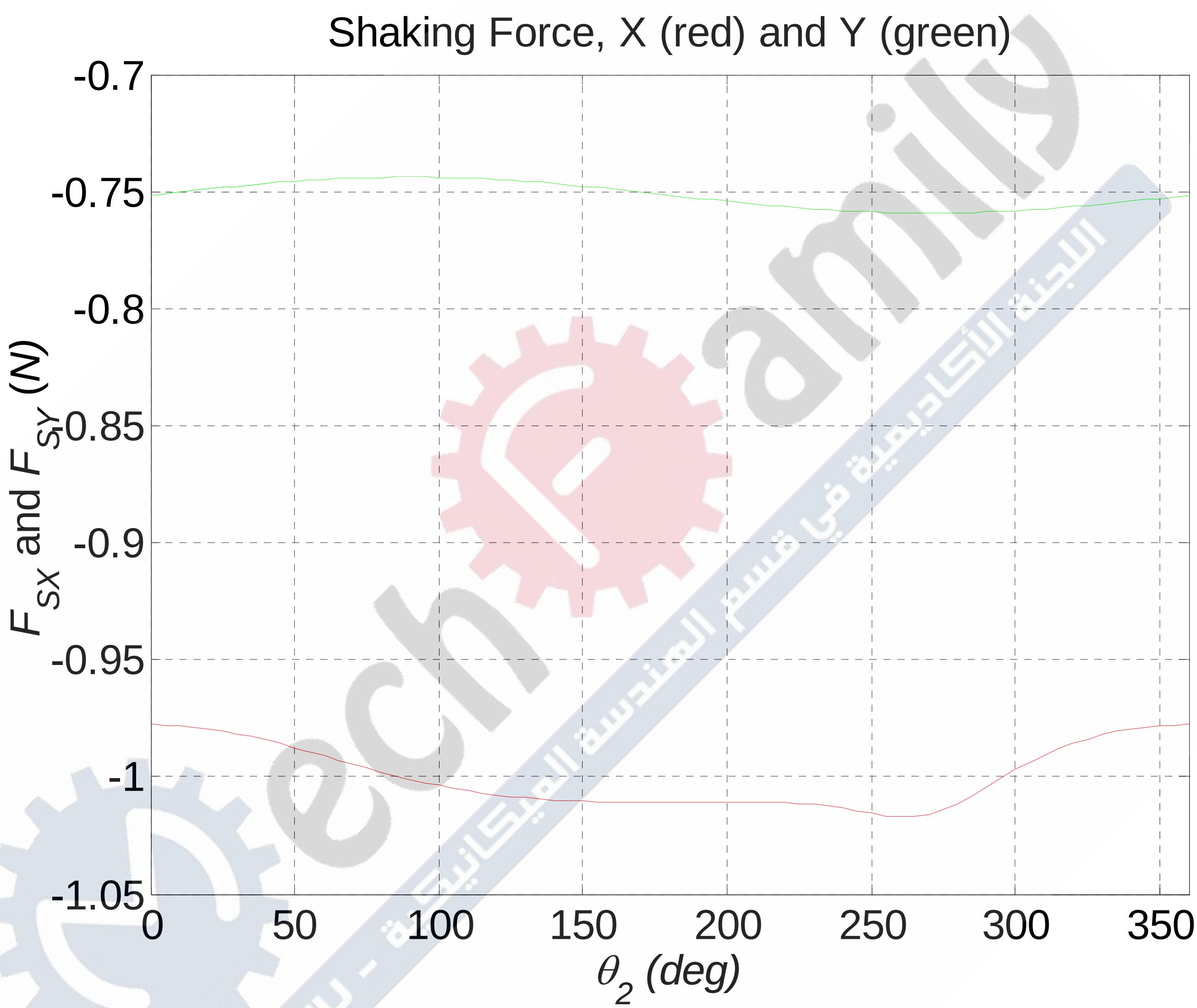
$$\underline{F}_S = \begin{Bmatrix} -0.982 \\ -0.748 \end{Bmatrix} \ (N) \qquad \underline{M}_S = -0.053\hat{k} \ \ (Nm)$$

<u>Full-Range-Of-Motion (F.R.O.M.) Analysis</u>

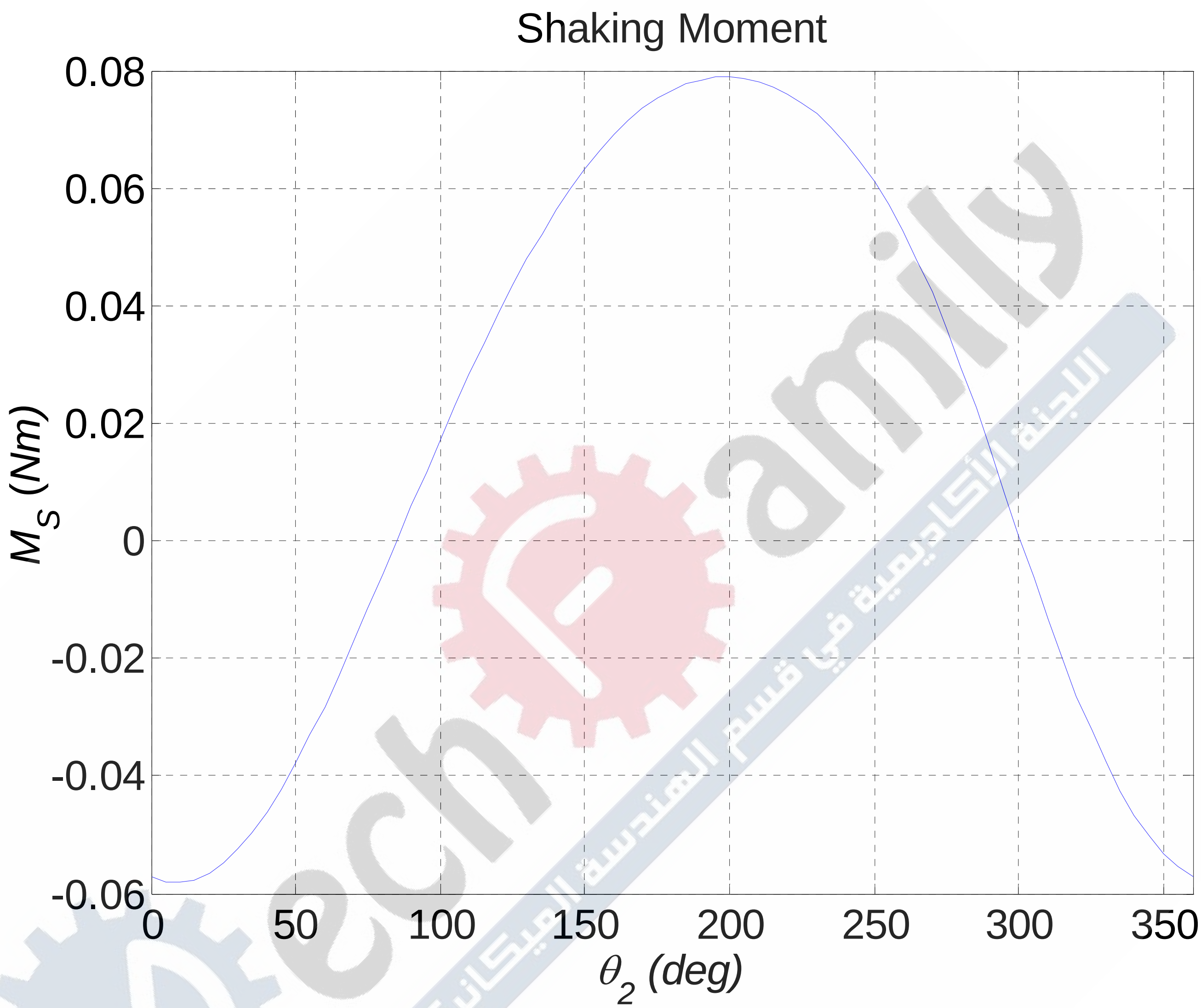
A more meaningful result from inverse dynamics analysis is to report the unknowns for the entire range of mechanism motion. The plot below gives the required driving torque $\underline{\tau}_2$ (Nm) for all $0^\circ \le \theta_2 \le 360^\circ$, for the Term Example 2 slider-crank mechanism, right branch only, assuming the given $\omega_2 = \pi/2$ rad/s is constant. This plot shows the torque (red) that must be supplied in all configurations by an external DC servomotor to cause the specified motion. Also plotted is the average torque (green) τ_{2AVG} = -0.002 and the root-mean-square torque value (blue) τ_{2RMS} = 0.086 Nm.



The plots below give the shaking force $\underline{F}_S$ (N) and shaking moment $\underline{M}_S$ (Nm) results, respectively, for all $0^\circ \le \theta_2 \le 360^\circ$. In the force plot, the X component is red and the Y green.



In the shaking moment plot, there is only the *Z* component:

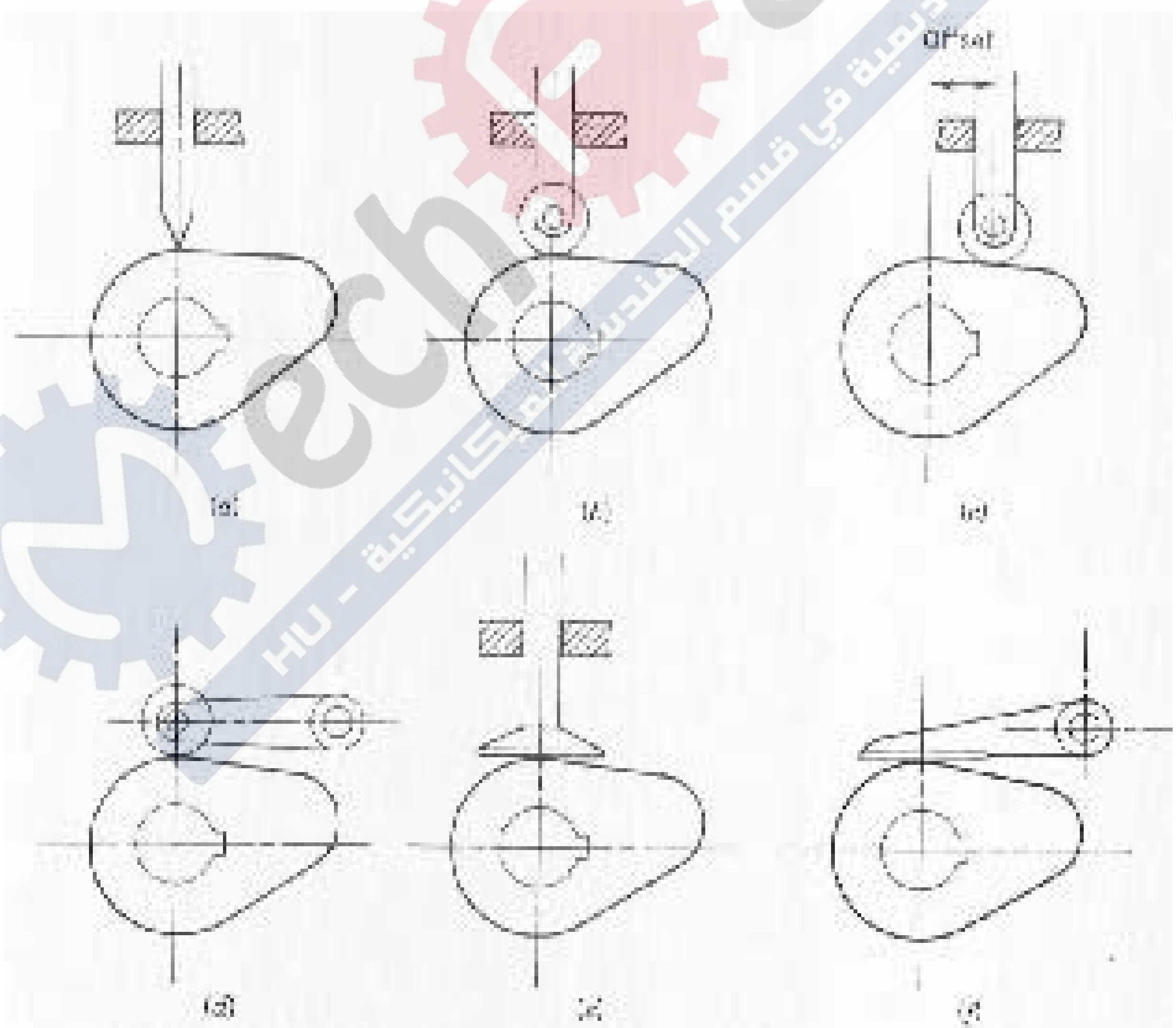


Cam Introduction

Chapter 8
Applications

Compared to linkages, easier to design desired motion with cams, but much more expensive and difficult to produce.

Cam Classification: Disk cams with followers



Degrees of Freedom Recall a cam joint has two-dof; allows both rolling and sliding.

Function Generation

The output parameter is a continuous function of the input parameter. With linkages, we can only satisfy a function exactly at a finite number of points: 3, 4, or 5, usually. For example, a 4-bar linkage:

$$\theta_4 = f(\theta_2)$$

With a cam and follower mechanism, we can satisfy function generation at infinite points.

$$S = f(\theta) \qquad \phi = f(\theta)$$

Cam input angle is θ , output is S for reciprocating (translating) and ϕ (rotating) for oscillating follower.

Cam Motion Profiles

Up to this point, we have been mostly concerned with mechanism ***analysis***: given a mechanism design and its input parameters, determine the position, velocity, acceleration, and dynamics behavior. With cams we must consider mechanism ***synthesis*** for the first time: given the motion requirements (follower motion and timing with input angle), design the cam. The first step is to determine a "smooth" cam follower motion profile. Classification:

When the motion transitions between different motion functions, we must ensure "smooth" motion.

Fundamental Law of Cam Design:

Which means:

If the *Fundamental law of Cam Design* is satisfied, the resulting dynamic performance will be acceptable for high-speed cam/follower operation. If not, there will be performance degradation due to noise, vibrations, high wear, etc. Cyclical impulse hammering when acceleration is not continuous.

***S V A J* Diagrams**

In synthesis, we are only given total motion range and perhaps some timing requirements. It is the engineer's job to determine the position curves and to match the velocity and acceleration across junctions. Position is automatically matched by shifting axes. Draw *S V A J* diagrams vs. time to graphically see if the *Fundamental Law of Cam Design* is satisfied for candidate curves. We can plot vs. time or vs. input cam angle θ (assuming constant angular velocity, $\theta = \omega t$).

Check out **Examples** **8-1** (terrible)
8-2 (bad)
8-3 (acceptable)

Slope of a function is the value of its derivative at a point. Therefore, for continuous velocity and acceleration curves, the slopes of the position and velocity curves must match across all junctions. The slope of the acceleration can be discontinuous (leading to finite jumps in jerk), but the acceleration itself must be continuous.

Generic Cam Follower Motion Profile figure:

Define each separate function so the value is zero at the initial angle, which is zero. Then to put the whole thing together, just shift the θ and S axes.

Match *S*: easy, just do it - shift S axes.

Match *V*: slope of S must match across junctions.

Match *A*: slope of V must match across junctions.

Cam Follower Motion Profile Examples **Example 1**

rise – dwell portion. Specify Parabolic (constant acceleration) to Straight Line (constant velocity) *rise*, followed by a *dwell*.

S: $f_1(\theta_1) = \frac{1}{2} A_0 \theta_1^2 \qquad f_2(\theta_2) = V_0 \theta_2 \qquad f_3(\theta_3) = 0$

V:

A:

J:

Match ***S*** at junction *B*: Just shift axis up.

Match ***V*** at junction *B*:

Try to match ***A*** at junction *B*:

Plot on next page.

Example 1 Plots

Cam Follower Motion Profile Examples   Example 2

Fix *rise* portion only. Specify Half-Cycloidal function (sinusoidal in cam angle) to Straight Line (constant velocity) *rise*.

S: $f_1(\theta_1) = L_1\left(\frac{\theta_1}{\beta_1} - \frac{1}{\pi}\sin\frac{\pi\theta_1}{\beta_1}\right) \quad f_2(\theta_2) = V_0\theta_2$

V:

A:

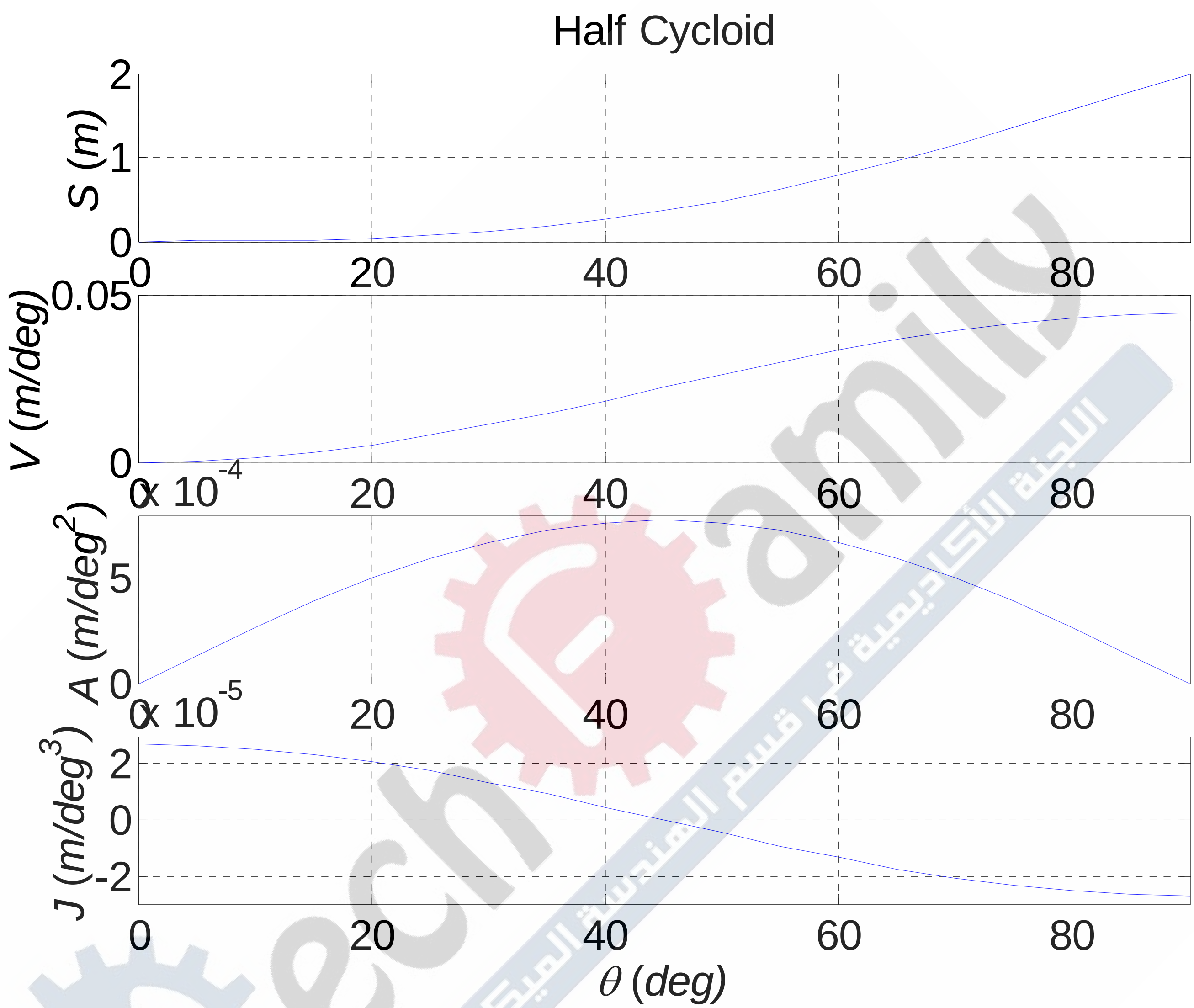
J:

Match ***S*** at junction *B*: Just shift axis up.

Match ***V*** at junction *B*:

Match ***A*** at junction *B*:

Plot on next page.


Half Cycloid
S (m)
V (m/deg)
x 10^-4
A (m/deg^2)
x 10^-5
J (m/deg^3)
θ (deg)
0
20
40
60
80
2
1
0
0.05
5
-2

<u>Cam Follower Motion Profile Examples</u> Example 3

Specify Full-Cycloidal function (sinusoidal in cam angle). This will *rise* all the way to meet a *dwell* smoothly; it satisfies the *Fundamental Law of Cam Design*.

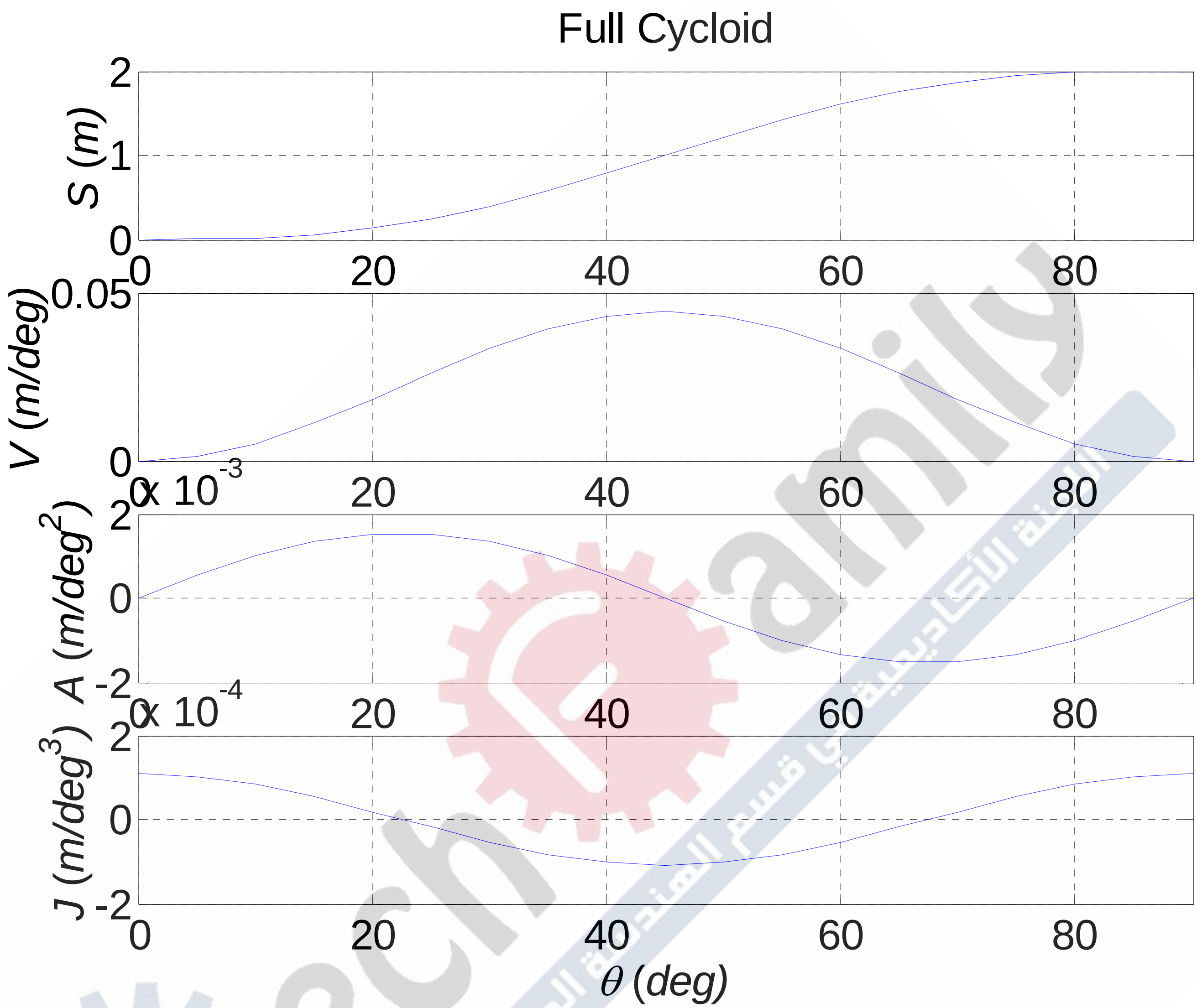
S: $f_1(\theta_1) = L_1\left(\frac{\theta_1}{\beta_1} - \frac{1}{2\pi}\sin\frac{2\pi\theta_1}{\beta_1}\right)$ $\quad f_2(\theta_2) = 0$

V: $v_1(\theta_1) = \frac{L_1}{\beta_1}\left(1 - \cos\frac{2\pi\theta_1}{\beta_1}\right)$ $\quad v_2(\theta_2) = 0$

A: $a_1(\theta_1) = \frac{2\pi L_1}{\beta_1^2}\left(\sin\frac{2\pi\theta_1}{\beta_1}\right)$ $\quad a_2(\theta_2) = 0$

J: $j_1(\theta_1) = \frac{4\pi^2 L_1}{\beta_1^3}\left(\cos\frac{2\pi\theta_1}{\beta_1}\right)$ $\quad j_2(\theta_2) = 0$

Plot on next page.



Full Cycloid
S (m)
V (m/deg)
x 10^-3
A (m/deg^2)
x 10^-4
J (m/deg^3)
θ (deg)

Analytical Cam Synthesis

Disk Cam with Radial Flat-Faced Follower

Assume a valid cam motion profile has been designed according to the *Fundamental Law of Cam Design*; i.e. we now have continuous ***S***, ***V***, ***A*** curves. Given the motion profile, determine the cam contour.

Is it as simple as polar plotting of $S = f(\theta)$ vs. cam angle θ?

We will use *kinematic inversion* to simplify the synthesis.

DCRFFF Figure:

As seen in the figure, the radius R out to the flat-faced follower (not to the point of contact (x,y)) is:

where C is the minimum cam radius, a design variable, and $S = f(\theta)$ is the given motion profile. The radius R and the flat-face length L can be related to the contact point x,y and the cam angle through geometry:

Notice that:

To calculate the follower flat-face length, double the maximum of L from above. Doubled because by symmetry the contact point will change to the other side at $\theta = 180^\circ$.

To summarize thus far:

This is sufficient to manufacture the cam; it is machined with θ, R, L coordinates. If we want to know the cam contour in Cartesian coordinates, we must solve the relationships for x, y. In matrix form:

The coefficient matrix $[A]$ is orthonormal, which means $[A]^{-1} = [A]^T$. The solution is:

Minimum Cam radius to Avoid Cusps

A cusp is a point in the cam, or actually undercut; this is to be avoided for good motion. The condition is that for a finite $\Delta\theta$, there is no change in x, y:

$$\frac{dx}{d\theta} = \frac{dy}{d\theta} = 0$$ will cause a cusp.

$$\frac{dx}{d\theta} = -(C + f(\theta))\sin\theta + \frac{df}{d\theta}\cos\theta - \frac{df}{d\theta}\cos\theta - \frac{d^2 f}{d\theta^2}\sin\theta$$

$$\frac{dy}{d\theta} = (C + f(\theta))\cos\theta + \frac{df}{d\theta}\sin\theta - \frac{df}{d\theta}\sin\theta + \frac{d^2 f}{d\theta^2}\cos\theta$$

$$\frac{dx}{d\theta} = -\left(C + f(\theta) + \frac{d^2 f}{d\theta^2}\right)\sin\theta$$

$$\frac{dy}{d\theta} = \left(C + f(\theta) + \frac{d^2 f}{d\theta^2}\right)\cos\theta$$

$\frac{dx}{d\theta} = \frac{dy}{d\theta} = 0$ simultaneously only when:

$$C + f(\theta) + \frac{d^2 f}{d\theta^2} = 0$$

Therefore, to avoid cusps on the entire cam contour,

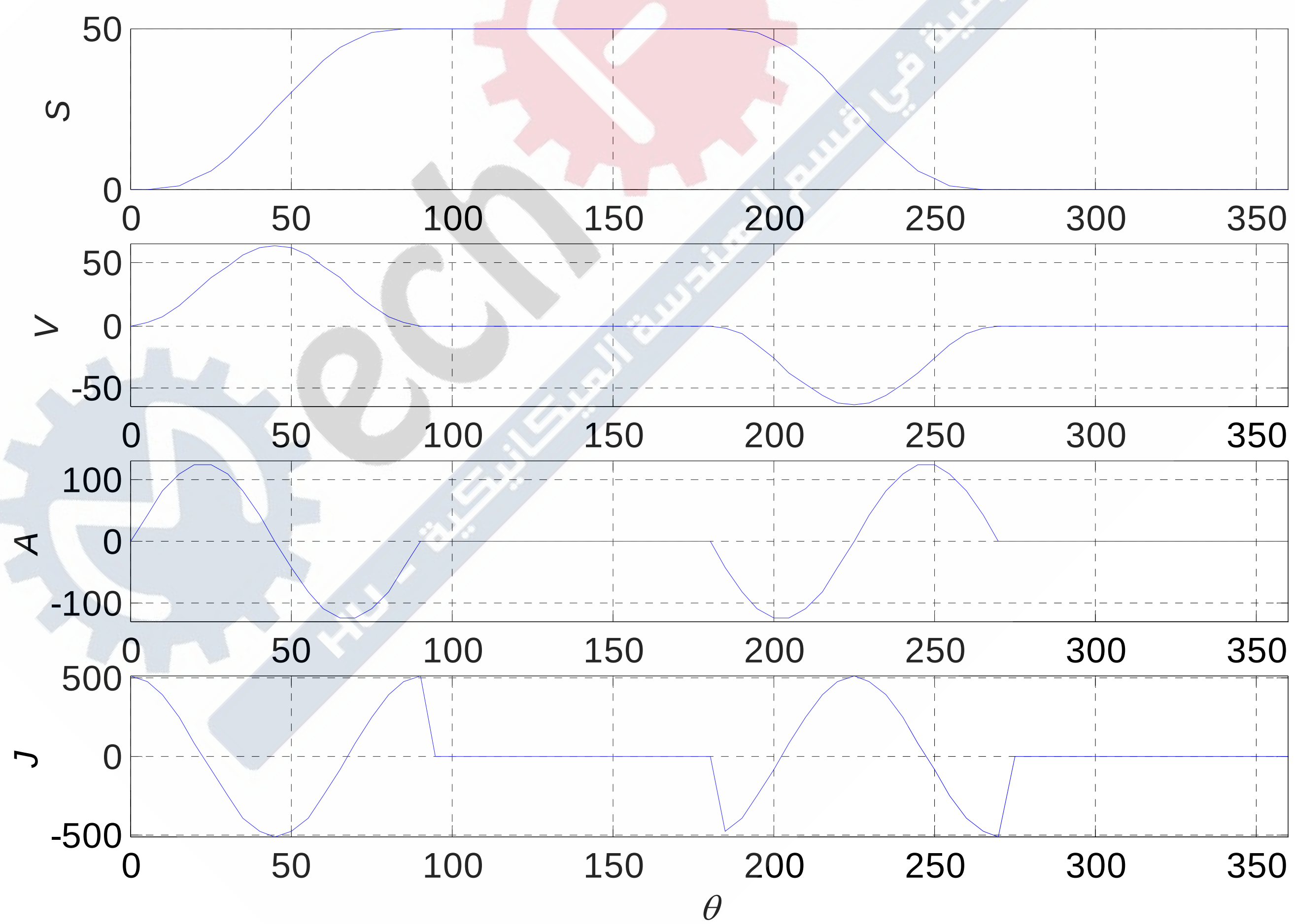
$$C + f(\theta) + \frac{d^2 f}{d\theta^2} > 0$$

Note C is always positive and $f(\theta)$ starts and ends at zero and never goes negative.

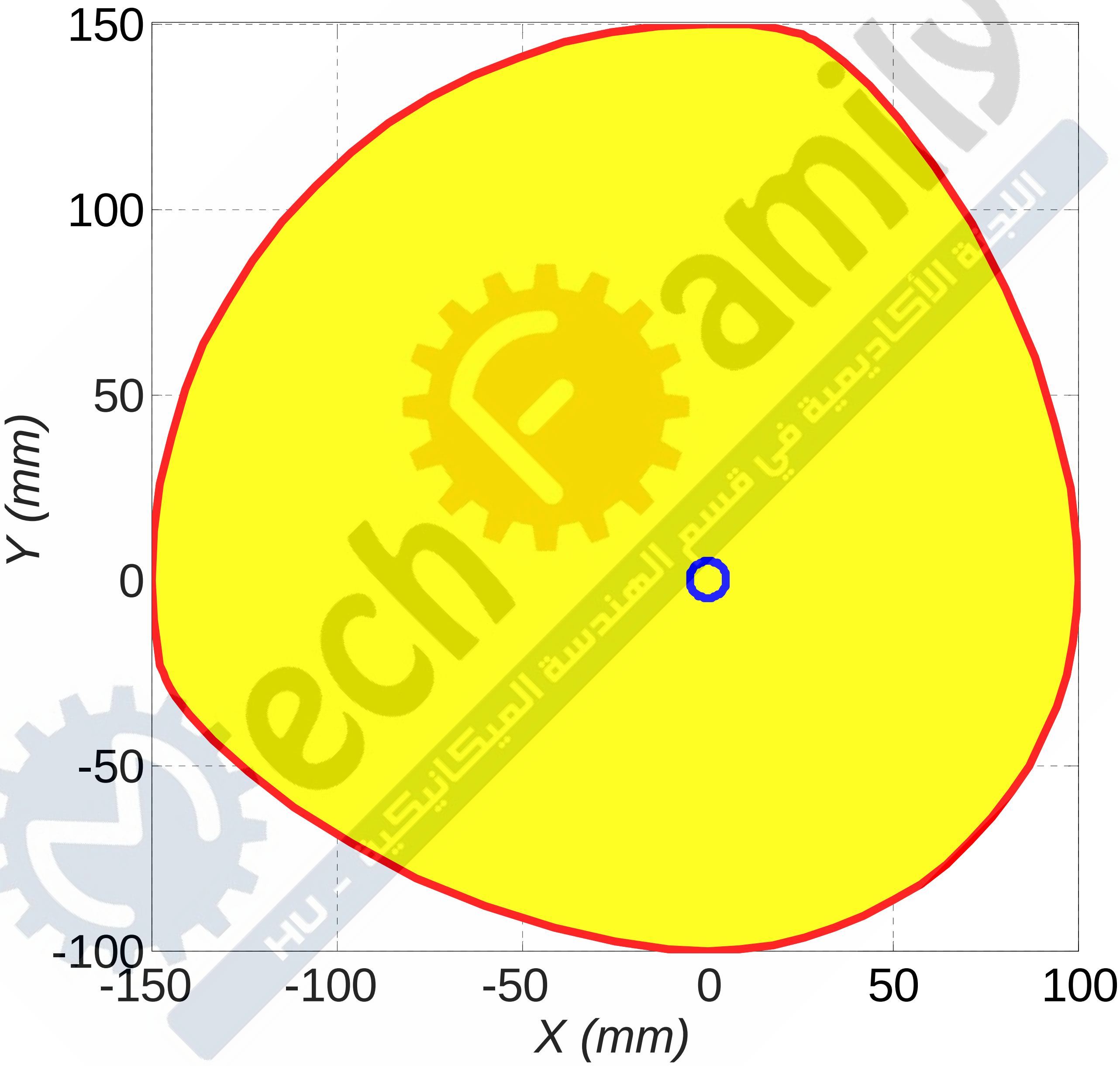
Disk Cam with Radial Flat-Faced Follower Design Example

Specify a full-cycloidal rise (total lift 50 mm), followed by a high dwell, a full-cycloidal return (total fall 50 mm), and then a low dwell. Each of these four motion steps occurs for 90 *deg* of cam shaft rotation.

The **cam motion profile** associated with this specification is shown below. Clearly, this satisfies the *Fundamental Law of Cam Design* because the position, velocity, and acceleration curves are continuous. The jerk is not continuous, but it remains finite over all cam angles.



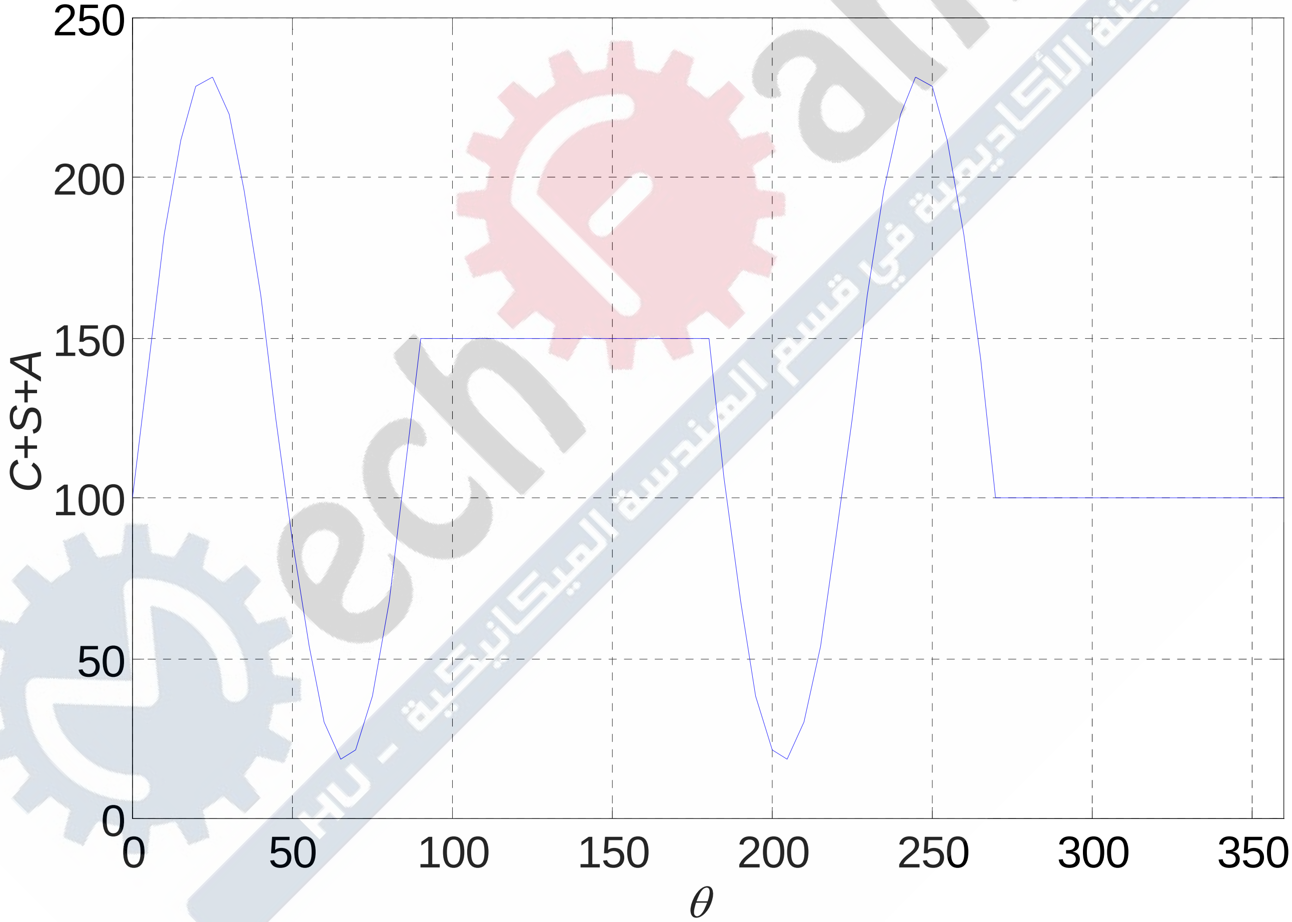
Choosing a minimum cam radius of $C = 100$ *mm*), the resulting **cam contour** is shown below.



Let us check the **cusp avoidance plot**. To avoid cusps in this cam, we require that:

$$C + S(\theta) + A(\theta) = C + f(\theta) + \frac{d^2 f}{d\theta^2} > 0$$

As seen in the plot below, this inequality is satisfied for the entire range of motion, so this cam design is acceptable.

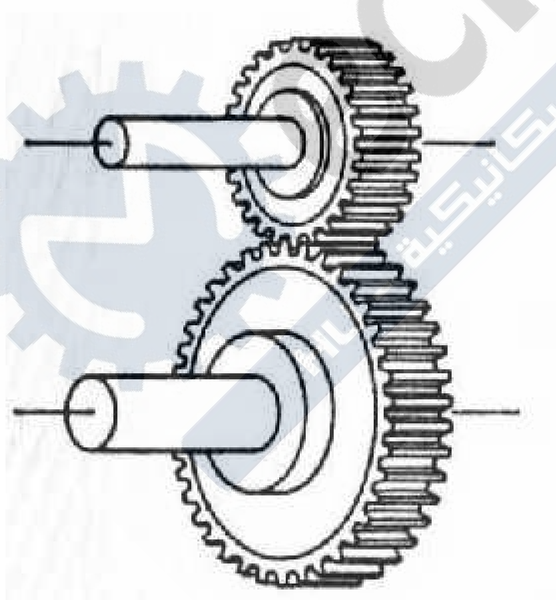


Gear Introduction

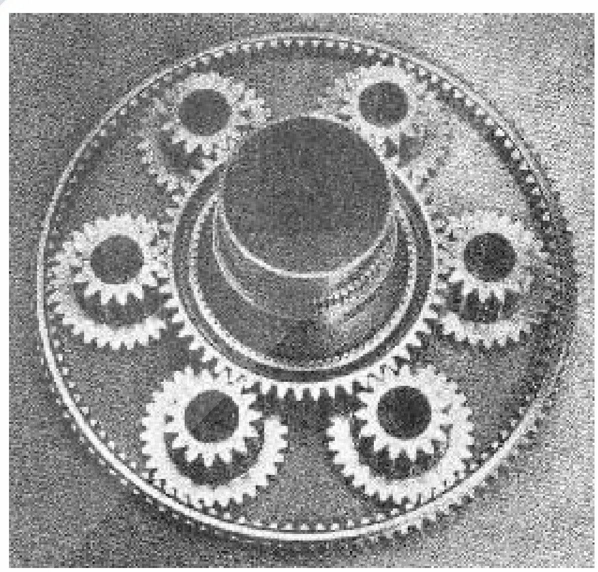
Transfer motion between rotating shafts in machinery, vehicles, toys, etc. Gears used in electromechanical systems. Change in angular velocity, torque, direction. Can openers to Aircraft carriers. Related mechanisms - belt and chain drives.

Applications:

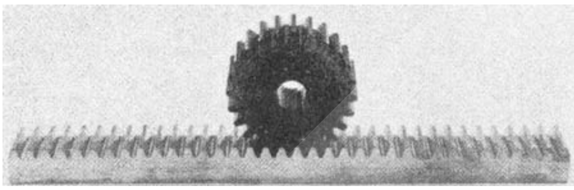
Gear Classification



External Spur Gears

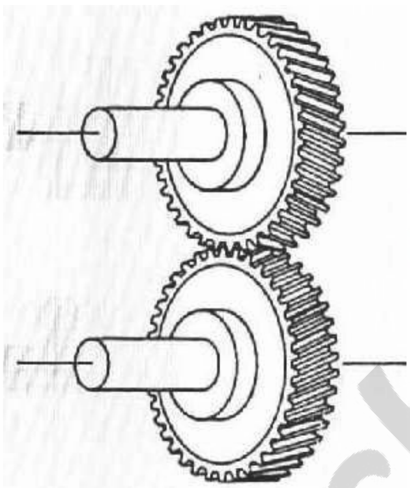


Internal Spur Gears

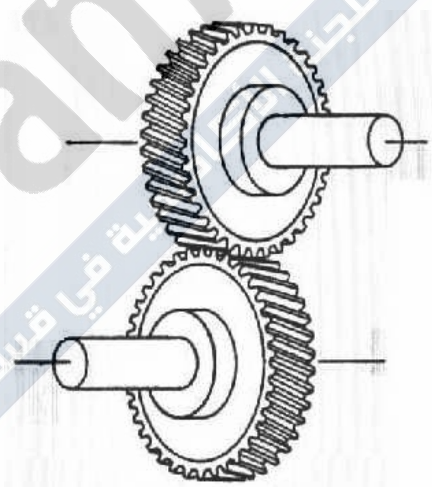


Rack & Pinion

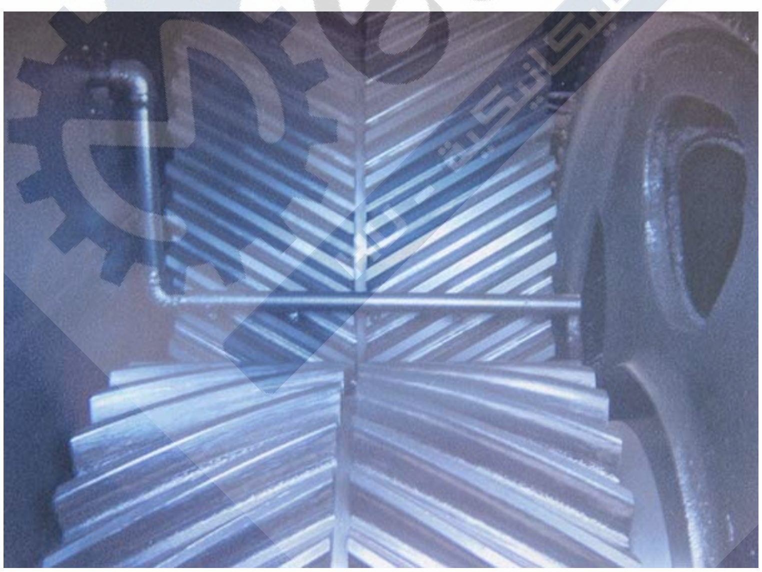
Bevel gears



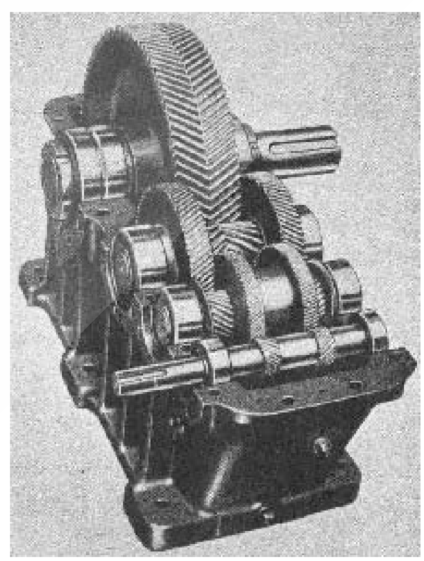
Helical (Parallel Shaft)



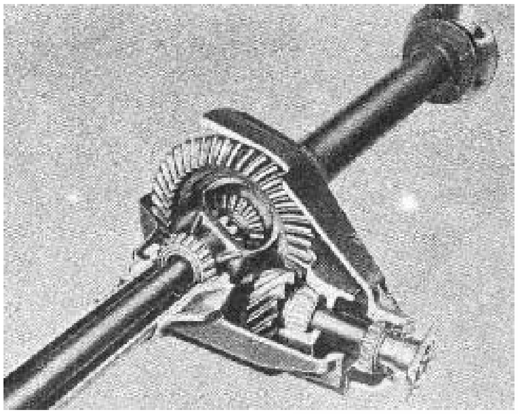
Helical (Crossed Shaft)



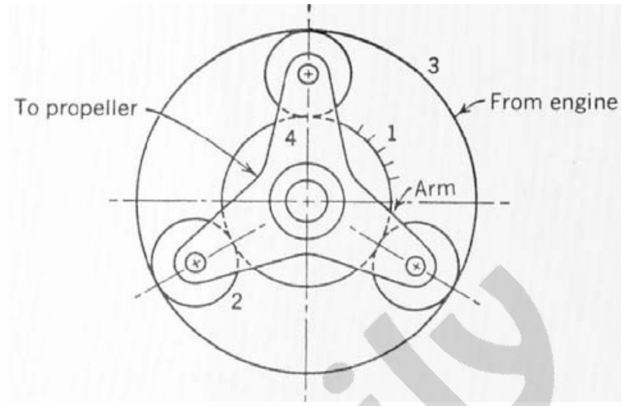
Herringbone Gears



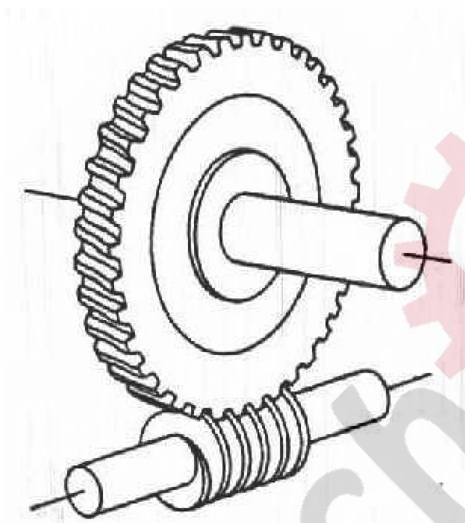
Gear Train



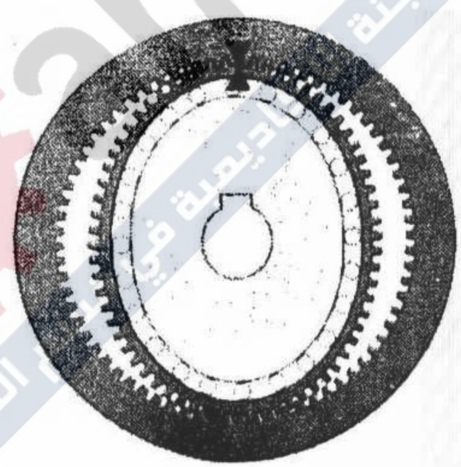
Automotive Differential



Planetary Gear Train



Worm and Gear



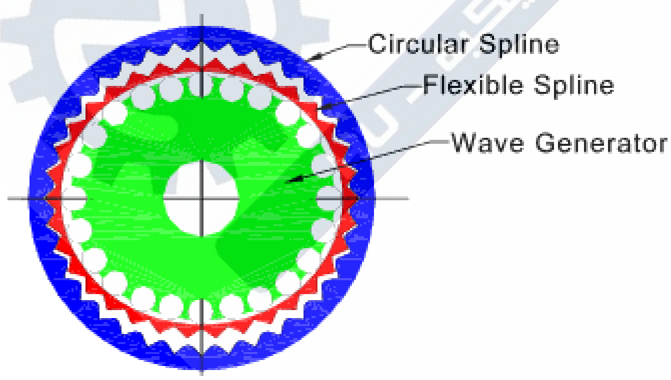
Harmonic Gearing

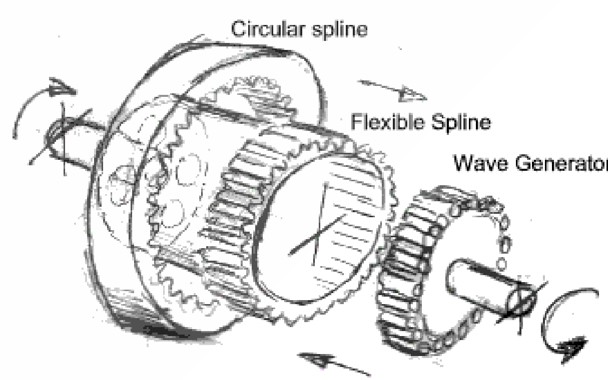
Harmonic Gearing

Taken from: http://www.roymech.co.uk/

"The harmonic gear allows high reduction ratios with concentric shafts and with very low backlash and vibration. It is based on a very simple construction utilising metals elasto-mechanical property."

"Harmonic drive transmissions are noted for their ability to reduce backlash in a motion control system. How they work is through the use of a thin-walled flexible cup with external splines on it lip, placed inside a circular thick-walled rigid ring machined with internal splines. The external flexible spline has two fewer teeth than the internal circular spline. An elliptical cam enclosed in an antifriction ball bearing assembly is mounted inside the flexible cup and forces the flexible cup splines to push deeply into the rigid ring at two opposite points while rotating. The two contact points rotate at a speed governed be the difference in the number of teeth on the two splines This method basically preloads the teeth, which reduces backlash."





Gear Ratio

Common electric motors have high speed but low torque. A robot joint needs lower rotation speed but high torque. A gear train can accomplish both objectives – reduce speed and increase torque. Gear ratio is a measure of the degree of reduction and increase.

Simple spur gear pair:

DOF:

A gear joint is like a cam joint; two-dof, teeth in contact allow rolling and sliding.

Gear 1 is input, gear 2 is output. Like two cylinders rolling without sliding. Arc lengths are equal.

Define gear ratio *n*:

Radii inversely proportional to angular motion. For standard spur gears, the radii are directly proportional to the number of teeth:

For relating angular velocities, tangential velocities are equal.

Most gear applications have constant angular velocities, for accelerating up to (or down from) constant angular velocities:

For relating shaft torques, interface forces are equal.

Radii directly proportional to shaft torques

Summary:

The ratio of the number of teeth is directly proportional to the radii, diameter, and shaft torques. The ratio of the number of teeth is inverse proportional to the shaft angles, angular velocities, and angular accelerations.

If $n > 1$: $\omega_2 < \omega_1$ Output has reduced speed
$\tau_2 > \tau_1$ Output has increased torque

This is the electric motor / robot joint case.

If $n < 1$: $\omega_2 > \omega_1$ Output has increased speed
$\tau_2 < \tau_1$ Output has reduced torque

If $n = 1$: $\omega_2 = \omega_1$ Output speed and torque unchanged
$\tau_2 = \tau_1$ direction reverses (external spur gears)

Gear ratio examples **<u>1. Gear toy</u>**

2. Mountain Bike Transmission

Gear Ratios: $n = \dfrac{N_{OUT}}{N_{IN}} = \dfrac{N_R}{N_F} = \dfrac{\omega_F}{\omega_R} = \dfrac{\tau_R}{\tau_F}$

Schwinn Sierra		Front 48	38	28
	14	0.29	0.37	0.50
	16	0.33	0.42	0.57
Rear	18	0.38	0.47	0.64
	22	0.46	0.58	0.78
	26	0.54	0.68	0.93
	30	0.62	0.79	1.07

Unlike electric motor example, mountain bike gearing generally:

- increases angular velocity
- decreases torque

Cannondale M400		Front 44	32	22
	11	0.25	0.34	0.50
	12	0.27	0.38	0.54
Rear	14	0.32	0.44	0.64
	16	0.36	0.50	0.73
	18	0.41	0.56	0.82
	21	0.48	0.66	0.95
	24	0.54	0.75	1.09
	28	0.64	0.88	1.27
	32	0.73	1.00	1.45

BikeE recumbent	Front 34	Rear hub 1.2913:1	1:1	0.7:1
	11	0.25	0.32	0.46
	13	0.30	0.38	0.55
Rear	15	0.34	0.44	0.63
	18	0.41	0.53	0.76
	21	0.48	0.62	0.88
	24	0.55	0.70	1.01
	28	0.64	0.82	1.18

But considering the difference in wheel sizes (26” Cannondale, 20” BikeE), the effective BikeE high and low gear ratios are:
Stiff: 0.32 instead of 0.25
Granny 1.53 instead of 1.18

Original front was 46 teeth – changed for more granny gear.

Gear Trains and Gear Standardization

Simple Gear Trains

Mesh any number of spur gears. Leftmost is driving gear. Rightmost is the output gear. All intermediate gears are first the driven gear and then the driving gear as we proceed from left to right. Let us calculate the overall gear ratio.

$$n_{GT} = \frac{\omega_{IN}}{\omega_{OUT}}$$

Example:

We can find the overall gear ratio by canceling neighboring intermediate angular velocities:

Each term in the above product may be replaced by its known number of teeth ratio:

All intermediate ratios cancel, so:

We could have done the same with pitch radii instead of number of teeth because they are in direct proportion:

So, the intermediate gears are ***idlers***. Number of teeth effect cancels out, but do change direction! We should have included sign:

So, for external spur gear trains:

Odd # of gears:	Output same direction as input
Even # of gears:	Output opposite direction as input

Different case:

Mesh any number of spur gears, where the driving and driven gears are distinct, because each pair is rigidly attached to the same shaft. See the figure. Again, let us calculate the overall gear ratio.

$$n_{GT} = \frac{\omega_{IN}}{\omega_{OUT}}$$

Example:

Again, we use the equation:

But now the gears rigidly attached to the same shaft have the same angular velocity ratio, so:

General formula:

Again, we must consider direction:

So, for external spur gear trains:

Odd # of pairs:	Output opposite direction as input
Even # of pairs:	Output same direction as input

Involute Spur Gear Details and Standardization

Rolling Cylinders

Mating spur gears are based on two pitch circles rolling without slip. These are fictitious circles; you cannot look on a gear to see them. The actual gear teeth both roll and slide (two-dof joint).

Fundamental Law of Gearing:

From our study of linkage velocity, we know this is no easy feat. Velocity ratios in a linkage vary wildly over the range of motion.

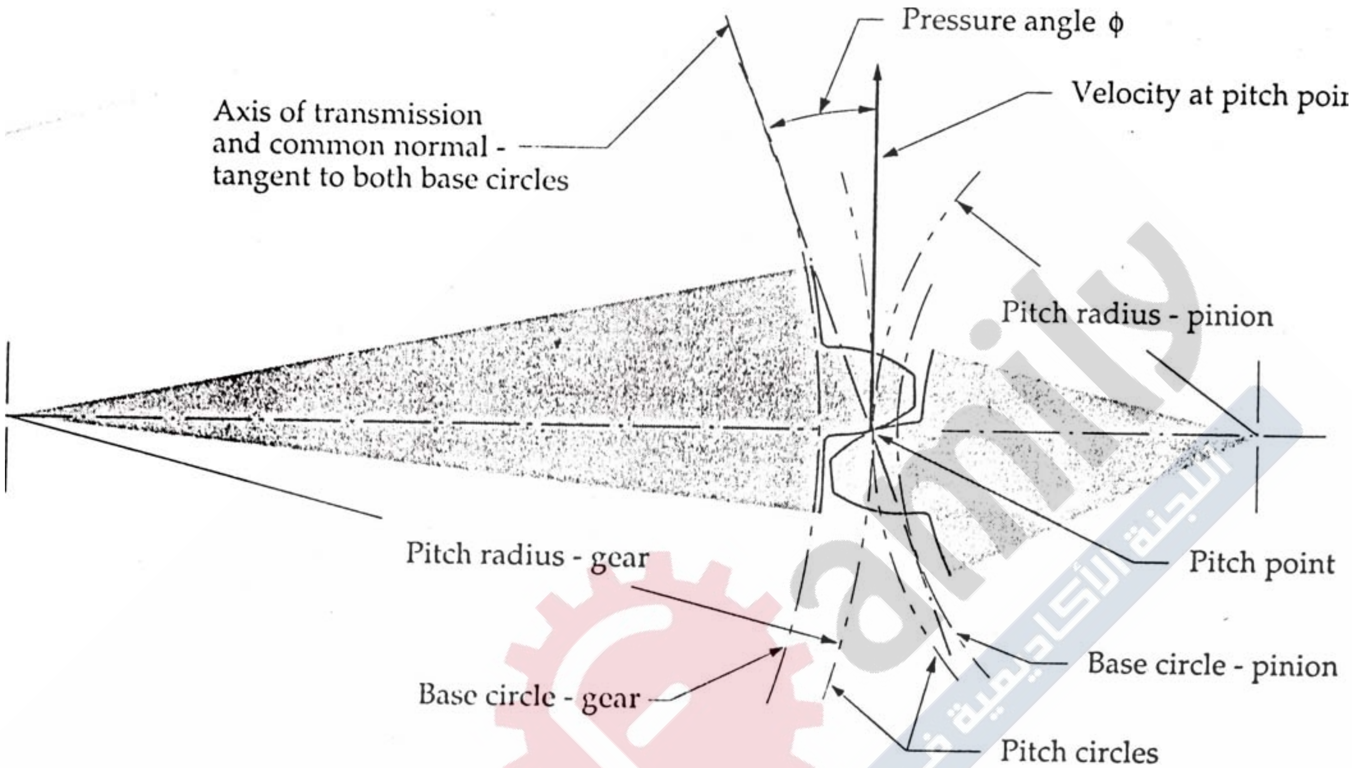
Velocity Ratio

Torque Ratio (Mechanical Advantage)

The author's velocity ratio is the inverse of our gear ratio definition and his torque ratio is the same as our gear ratio.

<u>Involute Function</u>

Standard spur gears have an involute tooth shape. If the gears' center distance is not perfect (tolerances, thermal expansion, wear, etc.), the angular velocity ratio will still be constant to satisfy the *Fundamental Law of Gearing*. The involute is a curve generated by unwrapping a taut sting from a circle:



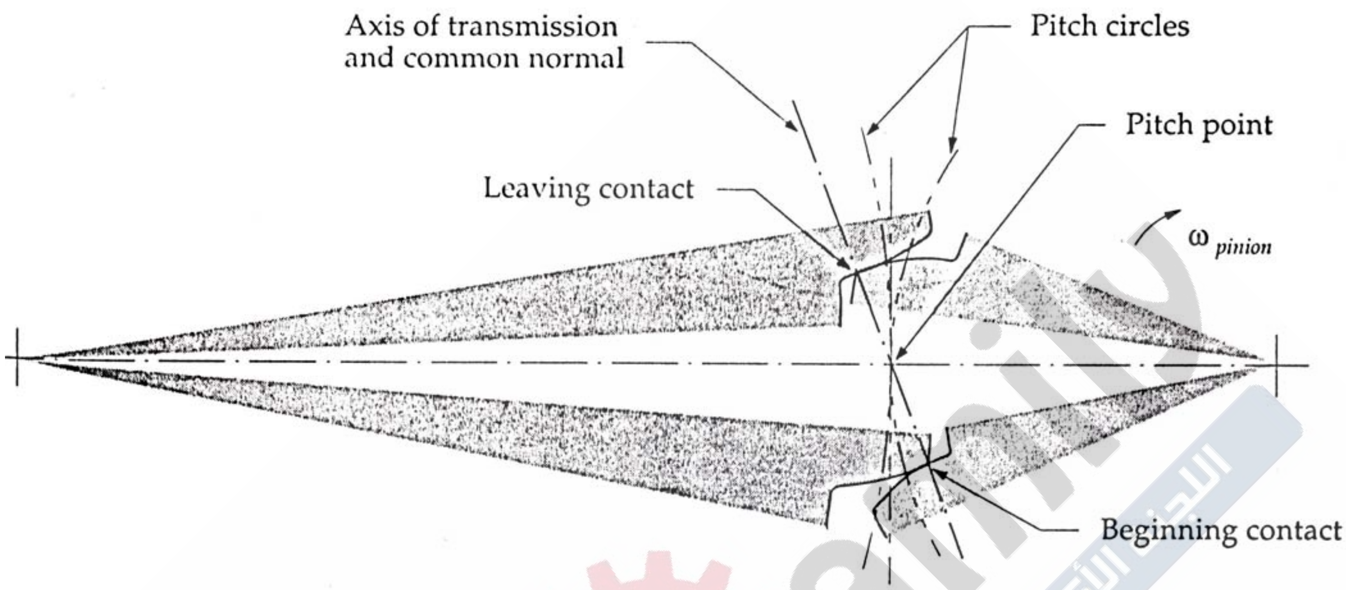
Base Circle: Involute starts from this circle

Pitch Circle: Fictitious circle, pure rolling in contact

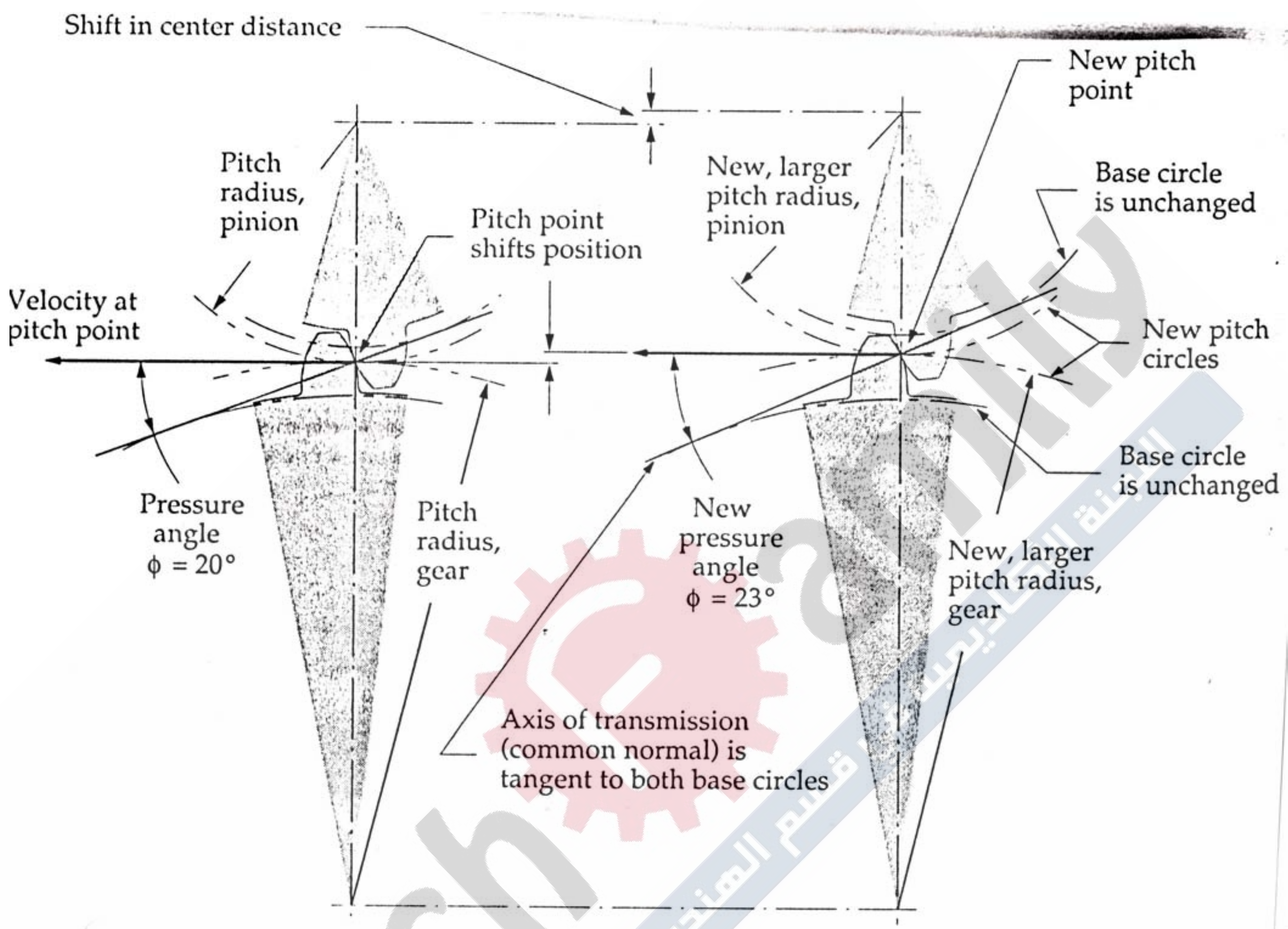
Pitch Point: Contact point between the two pitch circles

Pressure Angle: Angle between the common normal (also called axis of transmission) of the two meshing teeth and the velocity of the pitch point (tangent to both pitch circles). Point of contact slides along this line. Similar angle is defined for cams and followers.

Base circle, pitch circle, pressure angle relationship:



Length of contact along axis of transmission. Beginning of contact is when tip of driven gear tooth intersects the axis of transmission. End of contact is when tip of driving gear (pinion) tooth intersects the axis of transmission. Only one or two teeth are in contact at any one time. For harmonic gearing, many teeth are in contact at any one time (higher gear ratio in a smaller package).



Increasing center distance increases the pressure angle, increases the pitch circle radii, but doesn't change the base circles (of course). Thanks to the involute tooth shape, does not affect angular velocity ratio.

How is this possible? Relationship from last page:

Backlash

Clearance. Distance between mating teeth measured along the pitch circle circumference. All real gears must have some backlash due to tolerances, thermal expansion, wear, etc. However, must minimize backlash for smooth operation. Example: robot joints which must be driven both directions. Changing direction, nothing happens until the backlash is moved, and then impact - bad for dynamics. Non-linear effect in robots. On earth gravity tends to load the backlash for predictable effects. In space however, the backlash is less predictable! Figure:

Gear Standardization

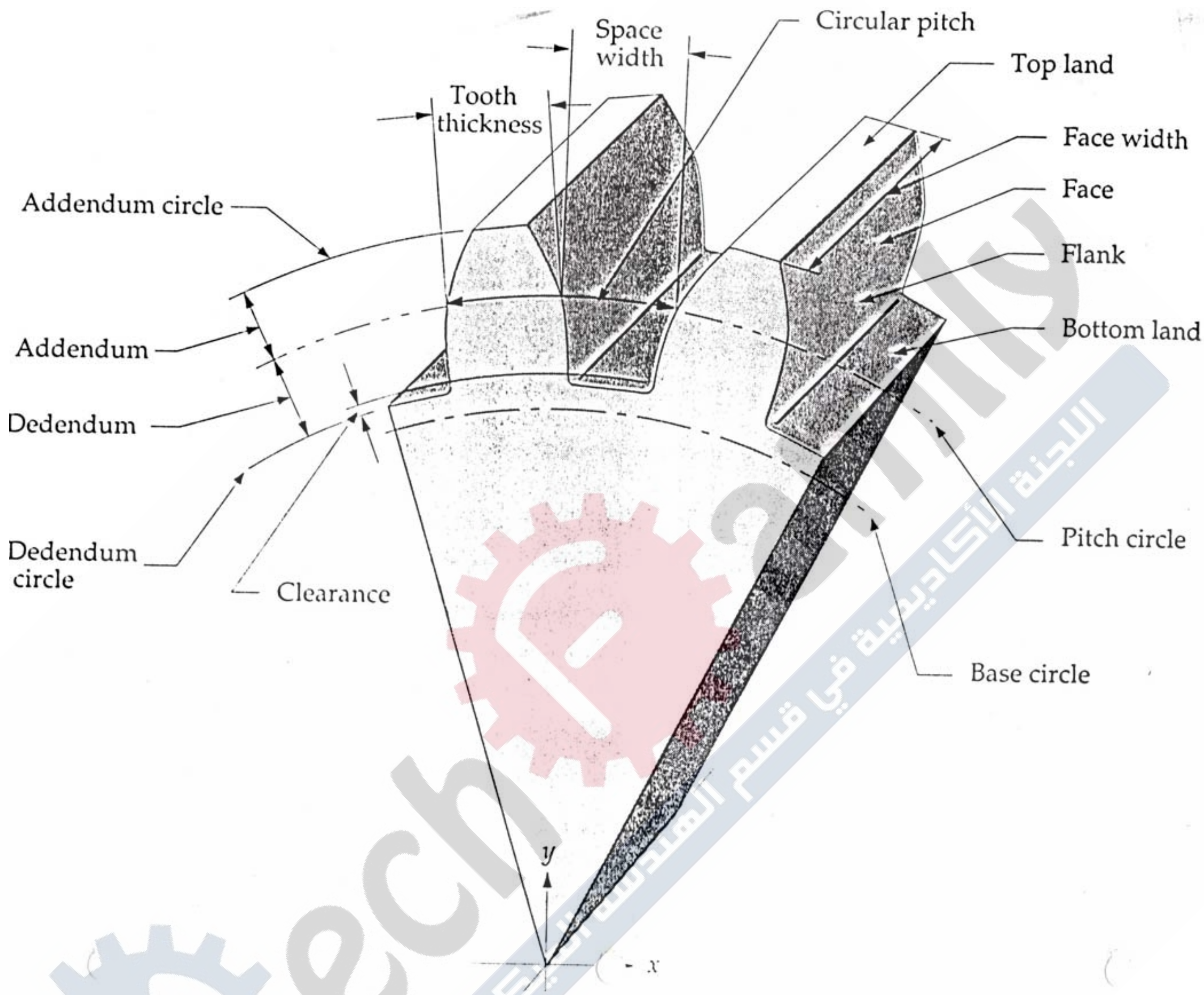
To allow interchangeability in manufacturing and to allow meshing of different size gears (radii and number of teeth) to achieve desired gear ratios. For two spur gears to mesh, must have: 1) the same *pressure angle*; 2) same *diametral pitch*; and 3) be made with *standard tooth proportions* (Table 9-1, p. 441).

Diametral Pitch:

Module:

Module is the metric version of diametral pitch. Not interchangeable with US gears because different tooth proportion standards!

Circular Pitch:



Standard involute tooth proportions, see Table 9-1, p. 441. Addendum is radial distance from pitch circle to Top Land of tooth. Dedendum is radial distance from pitch circle to Bottom Land of tooth (not to base circle). Clearance is radial distance from Bottom Land to mating gear Top Land (kinda like radial backlash). Face width is thickness of tooth and gear (mating widths needn't be the same). Tooth thickness is the circumferential length of each tooth.. Related to the circular pitch and backlash by: